BASUDEV GODABARI DEGREE COLLEGE, KESAIBAHAL

BLENDED LEARNING STUDY MATERIALS

UNIT-II

DEPARTMENT :-ECONOMICS

SUBJECT :-Mathematical Methods for Economics-II

SEMESTER :-2nd Semester

CONTENT

$f''(x) = 80x^3 + 36x$

$f'''(x) = 240x^2 + 36$

3. Partial Derivatives

Given a function y = f(x), the derivative f '(x), represents the rate of change of the function as x changes. For a function of two variables, such as z = f (x,y), one variable could be changing faster than the other variables. It will be completely possible for the function to be changing differently.

For a function of two independent variables, z = f(x,y), the partial derivative 'z' with respect to x may be found as normal rule of differentiation. The only difference is that, whenever or wherever the second independent variable 'y' appears, it will be treated as constant in every respect. Also the partial differentiation of y can be found by treating x variable as constant. Notations of partial differentiation are given below:

Notations of Partial Differentiation

Partial derivative of z w.r.t x	$\frac{\partial z}{\partial x}$	fx
Partial derivative of z w.r.t y	$\frac{\partial z}{\partial y}$	Fy

Example: $Z = x^4 y^2 - x^2 y^6$

$\frac{\partial z}{\partial x} = 4x^3y^2 - 2xy^6$

$\frac{\partial z}{\partial y} = 2x^4y - 6x^2 y^5$

Partial derivative can be defined as:-

If z = f(x,y) is a function of two variables, then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, called partial derivatives of z with respect to x and y respectively, be the derivative z w.r.t.x by keeping y as constant and the derivative z w.r.t.y by keeping x as constant. All the rules of differentiation can be applied when partial differentiation can be calculated.

Symbolically if f=f (x,y) then

$$fx = \frac{\partial f}{\partial x} = Lt_{\Delta x \to 0} \frac{f(x+\Delta x,y) - f(x,y)}{\Delta x}$$

$$fy = \frac{\partial f}{\partial y} = Lt_{\Delta y \to 0} \frac{f(x,y+\Delta y) - f(x,)y)}{\Delta y}$$

provided these limits exist.

Example 1. $z = f(x, y) = x^3y + x^2y^2 + xy + x + y^2$

$$\frac{\partial z}{\partial x} = 3x^2y + 2xy^2 + y + 1$$

$$\frac{\partial z}{\partial y} = x^3 + 2x^2y + x + 2y$$

3.1 Higher order partial derivative

For a function $z = f(x, y)$; $f'(x)$ & $f'(y)$ are the two first order partial derivatives with respect to x and y respectively. Since 'z' is a function hence $f'(x)$ and $f'(y)$ are also a function, hence, second order partial differentiation can also be found.

The second order partial derivatives are called mixed partial derivative because derivatives of more than one variable are to be observed. e.g differentiating a function with respect to 'x' first and then 'y' is called as mixed partial derivative. The various notation of partial derivative are given in table:

Notations of second order partial derivatives

Partial derivatives of z	Notation 1	Notation 2	Notation 3
w.r.t. x twice	$\frac{\partial}{\partial x}(\frac{\partial z}{\partial x})$	$\frac{\partial^2 z}{\partial x^2}$	fxx

w.r.t. y twice	$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$	$\frac{\partial^2 z}{\partial y^2}$	fyy
w.r.t. x first than y	$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$	$\frac{\partial^2 z}{\partial y\, \partial x}$	fxy
w.r.t. y y first than x	$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$	$\frac{\partial^2 z}{\partial x\, \partial y}$	fyx

A function has four possible second partial derivatives ones that are obtained by differentiating function w.r.t 'x' twice, w.r.t. y twice, w.r.t. x first than y and w.r.t. y first then x. All derivatives have sign (+ or -) interpretation of these signs are as follows.

Partial derivative	Sign	Interpretation
$\frac{\partial z}{\partial x}$	+ -	Slopes in x direction is positive Slopes in x direction is negative
$\frac{\partial^2 z}{\partial x^2}$	+ -	Slopes in x direction increases as x increases(y constant) Slopes in x direction decreases as x decreases(y constant)
$\frac{\partial z}{\partial y}$	+ -	Slopes in y direction is positive Slopes in y direction is negative
$\frac{\partial^2 z}{\partial y^2}$	+ -	Slopes in y direction increases as y increases(x constant) Slopes in y direction decreases as y decreases(x constant)
$\frac{\partial^2 z}{\partial y\, \partial x}$	+ -	Slopes in x direction increases as y increases(x constant) Slopes in x direction decreases as y decreases(x constant)
$\frac{d^2 z}{\partial x\, \partial y}$	+ -	Slopes in y direction increases as x increases(y constant) Slopes in y direction decreases as x decreases(y constant)

Example $z = x^{0.5}\, y^{0.5} - 60$ find $\frac{\partial z}{\partial x^2}$ interprate the result

sol. $\frac{\partial z}{\partial x} = 0.5\, x^{0.5}\, y^{0.5}$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(0.5\, x^{0.5}\, y^{0.5})$

$= (0.5)\,(-0.5\, x^{-0.5}\, y^{0.5})$

$= -\,0.25\, x^{-1.5}\, y^{0.5}$

Since x and y are positive, positive number raised to any power is positive; hence $y^{0.5}$ and $x^{-1.5}$ are positive , the term -0.25 in equation show that second order differentiation of z with respect to x twice is negative meaning that the slope in the x direction decreases as x increases when y is constant.

Example : $z = f(x, y) = x^3y + x^2\, y + 2x + xy + x + y^2$

$fx = x^3 + 2x^2y + 2xy^2 + y + 1$

$fy = x^3 + 2x^2y + x + 2y$

$fxx = 6xy + 2y^2$

$fyy = 2x^2 + 2$

$fxy = 3x^2 + 4xy + 1$

$fyx = 3x^2 + 4xy + 1$

.: $fxy = fyx$.

The two mixed second order partial derivatives (also called as cross partial derivatives) are always equal when fxy and fyx are continuous. It is explained by the following theorem given by Alexis Clairant also know as Young's theorem.

Theorem: Suppose f is defined on a disk D, which contains the point (a, b). If the partial derivatives fxy and fyx are both continuous on disk D, then
fxy (a, b) = fyx (a, b).

Example:- Verify Young's theorem $f(x,y) = x\, e^{-x^2y^2}$

Solution:-

$fx(x,y) = e^{-x^2y^2} - 2x^2y^2e^{-x^2y^2}$

$fy(x,y) = -2x^3y\,e^{-x^2y^2}$

Now, compute the two mixed partial derivatives.

$fxy(x,y) = -2x^2y\,e^{-x^2y^2} - 4x^2y\,e^{-x^2y^2} + 4x^4y^3e^{-x^2y^2}$

$= -6x^2y\,e^{-x^2y^2} + 4x^4y^3e^{-x^2y^2}$

$fyx(x,y) = -6x^2y\,e^{-x^2y^2} + 4x^4y^3e^{-x^2y^2}$

$\therefore fxy = fyx$

Hence proved.

3.2 Partial derivative with many variables

If $z = f(x_1, x_2 \ldots\ldots x_n)$ then

$\frac{\partial z}{\partial xi}$ is the differentiation of the function w.r.t. x_i when all the other variables x_j ($j \neq i$) are held constant.

i.e $\frac{\partial z}{\partial x_1} = \frac{\partial f(x1,x2\ldots\ldots xn)}{\partial x_1}$

and $\frac{\partial z}{\partial x_2} = \frac{\partial f(x1,x2\ldots\ldots xn)}{\partial x_2}$ and so on

Suppose, there is a function which consists of three variables $v = f(x, y, z)$. For such a function, there are partial derivatives of w.r.t x, y and z. When partial derivative has to take with respect to one of x, y and z assuming other two independent variables are constant.

In general, function consists of n variables. If $Z = f(x_1\ x_2, \ldots, x_n)$ then partial derivative of z w.r.t. x_i is $\frac{\partial z}{\partial xi}$ when all the other variables x_j ($j \neq i$) are held constant.

$$fx_i = \frac{\partial f}{\partial xi} = Lt_{h \to 0} \frac{f(x1\ x2,\ldots xi+h\ldots,xn) - f(x1\ x2,\ldots\ldots xi\ldots,xn)}{h}$$

provided limit exists.

Example:- $f(x, y, z) = x^2 + y^3 + z^4$

$fx = \frac{\partial (x,y,z)}{\partial x} = 2x$

$fy = 3y^2$

$fz = 4z^3$

Example: Find Zxxx, Zxyx, Zyyy,Zyxy of the function

$Z = 3x^2(5x+7y)$

$Zx = 3x^2(5) + (5x+7y)(6x)$

$= 45x^2 + 42xy$

$Zxx = 90x+42y$

$Zxxx = 90.$

$Zxy = 0 + 42x$

$= 42x$

$Zxyx = 42$

$Zy = 3x^2(7) + (5x+7y)(0)$

$= 21\, x^2$

$Zyy = 0$

$Zyyy = 0.$

$Zyx = 42x$

$Zyxy = 0.$

Example: Find Zxxx, Zxyy, Zyyy,Zxxy of the function $Z = (9x - 4y)(12x + 2y)$

$Zx = (9x - 4y)(12) + (12x + 2y)(9)$

$= 108x - 48y + 108x + 18y$

$= 216x - 30y$

$Zxx = 216$

$Zxyy = 0$

$Zxxx = 0$

$Zxy = -30$

$Zxyy = 0$

$Zy = (9x-4y)(2) + (12x+2y)(-4)$

$= 18x - 8y - 48x - 8y$

$= -30x - 16y$

$Zyy = -16$

$Zyyy = 0$

Example: Find Zxx, Zyy of the function $Z = \frac{x+y}{3y}$

$Zx = \frac{3y(1)-(x+y)(0)}{(3y)^2}$

$= \frac{1}{3y}$

$Zxx = 0$

$Zy = \frac{3y(1)-(x+y)(3)}{(3y)^2}$

$= \frac{-3x}{(3y)^2} = \frac{-x}{3y^2}$

$Zyy = -\frac{1}{3}(-2xy^{-3}) = \frac{2}{3}xy^{-3}$

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Clairaut theorem (Young's theorem) can be extended to any function of 'n' number of variables and their mixed partial derivations. The only thing has to remember that in each derivative, we differentiate with respect to each variable the same number of times.

For three variables, according to clairaut theorem,

$$f_{xz}(x, y, z) = f_{zx}(x,y,z)$$

provided with the derivatives are continuous.

The partial derivative is approximate equal to the change in function.i.e,

$f_i(x_1,\ldots x_n) \approx f_i(x_1,\ldots,x_{i-1}, x_{i+h}, x_{i+1}\ldots x_n) - f_i(x_1,\ldots,x_{i-1}, x_i, x_{i+1}\ldots x_n)$

There are n partial derivatives of first order. For each of the first order partial order derivative of the function, there are n second order derivatives. i.e.,

$$\frac{\partial}{\partial xj}\left(\frac{\partial f}{\partial xi}\right) = \frac{\partial^2 f}{\partial xj\,\partial x\,i} = fx_ix_j \quad (i=1..n;j=1..n)$$

So, total n^2 elements are there. Therefore, n*n matrix of second order partial derivative is the Hessian matrix which is symmetric and all $f_{11}=f_{22}=\ldots=f_n$(Clairant theorem)

$$\begin{bmatrix} f11 & f12\cdots & f1n \\ f21\vdots & f22\ddots & f2n\vdots \\ fn1 & fn2\cdots & fnn \end{bmatrix}$$

Example:

If the two demand functions for the two commodities are given by

$x=\frac{q}{p}$ $y=\frac{p2}{q}$

then the marginal demand functions are

$\frac{\partial x}{\partial p}=\frac{-q}{p2}$ $\frac{\partial y}{\partial p}=\frac{2p}{q}$

$\frac{\partial x}{\partial q}=\frac{1}{p}$ $\quad \frac{\partial y}{\partial q}=\frac{-p2}{q2}$

Since $\frac{\partial x}{\partial q}\geq 0$ and $\frac{\partial y}{\partial p}\geq 0$, therefore, two commodities are competitive.

Example:

If the demand functions for two related commodities are given by

$x=ae^{-pq}$ and $y=be^{p-q}$ where $a \geq 0 \quad b \geq 0$

Solution: Since two demand functions are given as

$$x=ae^{-pq}$$

$$y=be^{p-q}$$

their marginal demand functions can be calculated as:

$\frac{\partial x}{\partial p}= -aqe^{-pq}$ $\quad \frac{\partial y}{\partial p}= be^{p-q}$

$\frac{\partial x}{\partial q}= -ape^{-pq}$ $\quad \frac{\partial y}{\partial q}= -be^{p-q}$

Because $\frac{\partial x}{\partial q}\leq 0$ and $\frac{\partial y}{\partial p}\geq 0$, therefore the given commodities are neither competitive nor complementary.

Example

Consider two products, A and B. the demand for good A and B, & described by following two equations

$q_a = \frac{200}{pa\,pb/2}$

$q_b = \frac{1000}{p_a 1/3\, p_3}$ find $\partial q/ \partial q$ given the result explain A and Bare complementary or substitutes.

Solution

$$q_a = \frac{200}{pap^1b^2} = \frac{200\,pb^{-1/2}}{pa}$$

$$\frac{\partial qa}{\partial pb} = 200((\frac{-1}{2}p_b^{-1/2})/p_a)$$

$$= -100\frac{pb^{-3/2}}{pa}$$

$$=-100\,pa^{-1}\,pb^{-3/2}$$

$$q_b = \frac{1000}{pa^{1/3}pb}$$

$$\frac{\partial qb}{\partial pa} = \frac{\partial}{\partial pb} = (\frac{1000}{pa^{1/3}pb})$$

$$= \frac{1000}{pb}(\frac{\partial}{\partial pb}(p_a^{-1/3})$$

$$\frac{\partial q_b}{\partial pa} = \frac{\partial}{\partial pa}(\frac{1000}{pa^{1/3}\quad p_b})$$

$$= \frac{1000}{p_b}(\frac{-1}{3}p_a^{-1/3-1})$$

$$= \frac{-1000}{3}p_a^{-4/3}\cdot p_b^{-1}$$

We know that p_a and p_b are positive because prices can never be negative therefore:

$$\frac{\partial qa}{\partial pb} = -(\frac{100}{pa\,pb\,3/2}) = -(\frac{+}{++}) < 0$$

$$\frac{\partial qb}{\partial pa} = -\frac{1000}{3pa4/3\quad pb} = -(\frac{+}{++}) < 0$$

Because both cross elasticicities are – ve

Example: Two goods are complement goods.

The Stone-Geary Utility function is written as u= log U = β_1 log ($q_1 - \gamma_1$) + β_2($q_2 - \gamma_2$), where *u* is the utility index, q_i is the quantity of commodity *i*, $0 < \beta_i < 1$, $\gamma_i > 0$, $q_i - \gamma_i > 0$ and $i = 1,2$.

a. Find the marginal utility of this function with respect to q_1 and determine its sign.

b. What is the significance of a positive marginal utility?

c. Find the second derivative of this function with respect to q_1. Does the utility function exhibit diminishing marginal utility?

Solution: Utility function is: $u = \log U = \beta_1 \log(q_1 - \gamma_1) + \beta_2(q_2 - \gamma_2)$

a. Marginal utility is given by : $\frac{du}{dq_1} = \frac{\beta_1}{(q_1 - \gamma_1)}$; which is greater than zero; because both the numerator and the denominator are positive.

b. Since marginal utility is positive; this implies that as utility increases monotonically with increase in q_1.

c. $\frac{d^2u}{dq_1^2} = -\frac{\beta_1}{(q_1-\gamma_1)^2}$; which is less than zero. Since the second derivative is negative, the utility function exhibits diminishing marginal utility.

Example: Given the production function:

$$P(L,K)=5L^{1/2}K^{1/2}+L.$$

Find out the partial elasticity with respect to labor at (L,K)= (1024,27).

Solution: $P(L,K)=5L^{1/2}K^{1/2}+L$

$$\epsilon_L = P'_L(L,K).L\frac{L}{P(L,K)}$$

$$= (L^{-4/5}K^{-1/3}+1)\frac{L}{5L^{1/5}k^{1/3}+L}$$

$$=\frac{L^{1/5}k^{1/3}+L}{5L^{1/5}k^{1/3}+L}$$

Therefore, at (L,K)=(1024,27) we have $\epsilon_L = \frac{1024^{1/5}27^{1/3}+1024}{5.1024^{1/5}27^{1/3}+1024} = \frac{259}{271}$

This explains that if capital remains constant at K=27 and at L=24 labour increases with 1 percent, then output will increase by $\frac{259}{271}$ percent.

Example: Utility function is given:

$U = U = X^{0.5}Y^{0.5}$.

Calculate the marginal rate of substitution between X,Y.

Function is $U = X^{0.5}Y^{0.5}$.

First, take the partial derivative of U with respect to X to get MU_x.

$$MU_x = \frac{\partial U}{\partial x} = 0.5X^{-0.5}Y^{0.5}.$$

Next, take the partial derivative with respect to Y to get MU_y.

$$MU_y = \frac{\partial U}{\partial x} = 0.5X^{0.5}Y^{-0.5}.$$

Dividing MU_x by MU_y we get

$$MRS = \frac{MUx}{MUy} = \frac{0.5X^{-0.5}Y^{0.5}}{0.5X^{0.5}Y^{-0.5}} = y/x$$

Example: Given an isoquant

$$Q = K^{1/6}L^{1/2}$$

Find out slope of isoquant.

Solution: Slope of Isoquant $= \frac{dk}{dL}$

$$\frac{dk}{dL} = -\frac{\partial Q}{\partial L} / \frac{\partial Q}{\partial K}$$

$$\frac{\partial Q}{\partial L} = \frac{1}{2}K^{1/6}L^{-1/2}$$

$$\frac{\partial Q}{\partial K} = \frac{1}{6}K^{-5/6}L^{1/2}$$

$$\frac{dk}{dL} = -(\frac{1}{2}K^{1/6}L^{-1/2}) / (\frac{1}{6}K^{-5/6}L^{1/2})$$

$$= -\frac{6}{2}K^{1/6}K^{5/6}L^{-1/2}L^{-1/2}$$

$$= -3(K/L)$$

Therefore, the slope of isoquant is 3(K,L).

Example: Given demand function Q- 90+ 2P=0; and average cost function

$$AC = Q^2 - 39.5Q + 120 + 125/Q$$

Calculate the level of output where:

(a) total revenue is maximum,

(b) marginal cost is minimum,

(c) profits is maximum.

Solution: (a) The demand function is Q- 90+ 2P=0.

Written as

$$2P=90 - Q$$

$$P= 45 - 0.5Q$$

$$TR= PQ= (45 - 0.5Q)Q$$

$$=45Q - 0.5\ Q^2$$

For maximizing TR, first-order condition is:

$$\frac{dTR}{dQ}=45 - Q=0$$

$$Q=45$$

and second-order condition is, $\frac{d^2TR}{dQ^2}= -1<0$.

Therefore, at Q=45, TR is maximized.

(b)

$$AC= Q^2- 39.5Q+ 120+ 125/Q$$

$$TC= AC.Q= (Q^2- 39.5Q+ 120+ 125/Q)Q$$

$$= Q^3- 39.5Q^2+ 120Q+ 125$$

$$MC= \frac{dTC}{dQ} = 3Q^2 -79Q+ 120$$

MC is minimum when, $\frac{dMC}{dQ}=0$ and $\frac{d^2MC}{dQ^2}$

$$\frac{dMC}{dQ} = 6Q -79=0$$

$$Q= 13.167$$

$$\text{And, } \frac{d^2MC}{dQ^2}=6>0.$$

Hence, at Q= 13.167, MC is minimum.

(c)

$$\text{Profit } (\pi) = TR- TC$$

$$=45Q - 0.5\ Q^2 - (Q^3- 39.5Q^2+ 120Q+ 125)$$

$$= -Q^3 + 39\ Q^2 - 75Q- 125$$

For maximization of profit, first order condition

$$\frac{d\pi}{dQ} = - 3Q^2 + 78Q- 75=0$$

$$(-3Q+3)(Q-25)=0$$

$$Q=1 \text{ and } Q=25$$

and for second order condition,

$$\frac{d^2\pi}{dQ^2} = -6Q+78$$

When Q = 1 then,

$$\frac{d^2\pi}{dQ^2} = 72>0.$$

When Q=25 then,

$$\frac{d^2\pi}{dQ^2} = -72<0.$$

Therefore, profit is maximum when Q=25

Maximum $\pi = -(25)^3 + 39\,(25)^2 - 75(25) - 125 = 6750.$

Example: Two different demand functions are given:

$Q_1 = 21 - 0.1P_1$ and $Q_2 = 50 - 0.4\,P_2$

TC = 2000 + 10 Q where $Q = Q_1 + Q_2$, what price will the firm charge (a) with discrimination and (b) without discrimination between markets?

Solution: since demand function in first market is, $Q_1 = 21 - 0.1P_1$

Therefore, $P_1 = 210 - 10\,Q_1$

And, $TR_1 = P_1Q_1 = (210-10Q_1)Q_1 = 210Q_1 - 10\,Q_1^2$

$$MR_1 = \frac{dTR_1}{dQ_1} = 210 - 20Q_1$$

Profit is maximum when MR= MC,

$$MC = \frac{dTC}{dQ} = 10$$

$MR_1 = MC$

$210 - 20Q_1 = 10$

$Q_1 = 10$

When $Q_1 = 10$, $P_1 = 210 - 10(10) = 110$

demand function in second market is, $Q_2 = 50 - 0.4P_2$

hence, $P_2 = 125 - 2.5\,Q_2$

$TR_2 = (125 - 2.5\,Q_2)Q = 125Q - 2.5\,Q_2^2$

$MR_2 = \frac{dTR_2}{dQ_2} = 125 - 5Q_2$

When $MR_2 = MC$

$125 - 5Q_2 = 10$

$Q_2 = 23$

When $Q_2 = 23$, then $P_2 = 125 - 2.5(23) = 67.5$

The discriminating monopoly charges a lower price in the second market where the demand is relatively more elastic, and a higher price in the first market where the demand is relatively less elastic.

Example: A producer is a price-taker on both the market for input factors labor and capital, and the market for end products. The cost of one unit of labor equals w = 2, the cost of one unit of capital equals r = 32 , while the selling price of the end products equals p =32. The production function of this producer is given by $Y(L,K) = L^{1/8} K^{1/2}$. Determine the maximum profit.

Solution:

The revenue function is $R(L,K) = p.Y(L,K) = 32. L^{1/8} K^{1/2}$

Cost function

$C(L,K) = wL + rK = 2L+32K$, and

Hence, profit function becomes

$\Pi(L,K) = 32 L^{1/8} K^{1/2} - 2L - 32K$

Partial derivative of $\pi(L,K)$ is given by:

$\pi'_L = 4L^{-7/8}K^{1/2} - 2$ and

$\pi'_k = 16 L^{1/8}K^{-1/2} - 32$

the stationary points of profit function are solutions of the following system

$4L^{-7/8}K^{1/2} - 2 = 0$

$16 L^{1/8}K^{-1/2} - 32 = 0$

Hence, $K^{1/2} = 1/2 L^{7/8}$ and

therefore, $K = ¼ L^{14/8}$

Consequently, $L^{1/8}(1/4 L^{14/8})^{-1/2} = 1$

which gives L=1 and therefore, K=1/4

Hence, (L,K) = (1,1/4) is the only stationary point. By the use of the criterion function we investigate whether or not this point is a maximum location.

$\pi''_{LL} = -3.1/2L^{-15/8}K^{1/2}$;

π''_{KK}= -8L$^{1/8}$K$^{-3/2}$ and

π'_L= 2L$^{-7/8}$K$^{-1/2}$, which implies that the criterion function is given by

C(L,K) = π''_{LL}. π''_{KK} - $(\pi''_{LK})^2$

= (-3.1/2L$^{-15/8}$K$^{1/2}$)(-8L$^{1/8}$K$^{-3/2}$)- (2L$^{-7/8}$K$^{-1/2}$)^2

= 28 L$^{-14/3}$K^{-1} – 4L$^{-14/8}$K^{-1}

= 24 L$^{-14/8}$K^{-1}>0

Hence, as C(1,1/4) >0 and π''_{LL} (1,1/4) <0 it follows that $\pi(L,K)$ has a maximum profit at (L,K)= (1,1/4), with value π=6.

4. Quadratic Forms

A quadratic form of two variables is

f(x,y) = ax^2 + 2bxy +cy^2;

a,b,and c are constants. Now, using matrix notation:

$$f(x,y) = (x,y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

f''_{11}= 2a, f''_{12}= f''_{21}= 2b and f''_{22}= 2c are the second order partial derivatives of the function f(x,y) Therefore, the Hessian of f is given by

$$2\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

The given quadratic form is said to positive definite if f(x,y) >0; for all values of x and y i.e, (x,y) $\neq$ (0,0), and positive semidefinite if f(x,y)$\geq$ 0 for all values of (x,y). The given function is negative definite if f(x,y)<0; for all values of x and y; and it is negative semidefinite if f(x,y)$\leq$0. And it is indefinite we have two different pairs of x and y; (x$^-$,y$^-$) and (x$^+$,y$^+$); and also f(x$^+$,y$^+$) >0.

Example: Express the quadratic form below as a matrix form. Determine the definiteness of the equations:

a) f(x$_1$,x$_2$) = 4x^2 +8xy +5y^2

b) f(x$_1$,x$_2$) = -x^2 +xy - 3y^2

Solution: a) $f(x,y) = (x,y)\begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Therefore, symmetric matrix is $\begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}$, whose determinant is positive. Hence,f(x,y) > 0 for all values of x and y. Therefore, the quadratic form is positive definite.

$$f(x,y) = (x,y)\begin{pmatrix}-1 & 1/2\\ 1/2 & -3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

Therefore, symmetric matrix is $\begin{pmatrix}-1 & 1/2\\ 1/2 & -3\end{pmatrix}$, whose determinant is negative. Hence, $f(x,y) < 0$ for all values of x and y. Therefore, the quadratic form is negative definite.

5. Exercise:

1. Find the second – order partial derivatives fxx, fyy and fxy for each of the following functions:

   (a) $Z = \frac{x+4}{2x+5y}$

   (b) $Z = (7x + 3y)^3$

   (c) $Z = (x^3 + 5y)^5$

   (d) $Z = (2x+5y)e^y$

   (e) $Z = \log(1+x^2) + y^2$

   (f) $Z = 3x^2e^{2y}$

2. Consider the function: $f(x_1, x_2) = (3x_1^2 + 5x_1 + 1)\cdot(x_2 + 4)$.

a. Find f_1 and f_2.

b. Find f_{11}, f_{22}, f_{12} and f_{21}.

3. Assume the demand for sugar is a function of income (Y), the price of sugar (P_s) and the price of saccharine (P_c), a sugar substitute, as follows:

$Q_d = f(Y, P_c, P_s) = 0.05Y + 10P_c - 5P_s^2$.

a. Find the partial derivatives of this demand function.

b. Find the elasticity of demand with respect to income $\left(\frac{\partial Q_d}{\partial Y}\cdot\frac{Y}{Q_d}\right)$ when Y = 10,000, P_s = 5 and P_c = 7.

c. Find the own-price elasticity of demand $\left(\frac{\partial Q_d}{\partial P_s}\cdot\frac{P_s}{Q_d}\right)$ when Y = 10,000, P_s = 5 and P_c = 7.

d. Find the cross-price elasticity of demand $\left(\frac{\partial Q_d}{\partial P_c}\cdot\frac{P_c}{Q_d}\right)$ when Y = 10,000, P_s = 5 and P_c = 7.

4. Show that fxz = fzx and fxzz = fzxz = fzzx from the following function:

$F(x,y,z) = ye^{x} + x\log z$

5. The demand function of two related commodities are given by

$$X_1 = p_1^{-1.7} p_2^{0.8}$$

$$X_2 = p_1^{0.5} p_2^{-0.2}$$

What can you say about the two commodities X_1 and X_2 and also find all partial elasticties.

6. A firm produces two commodities; commodity X and commodity Y, the demand functions are:

$p_1 = 8 - 2x$

$p_2 = 14 - y^2$

The combined cost of production of these unit is given by $C = 10 + 4x + 2y$. What will be the prices of two products so that joint profit will be the maximum.

7. Consider a production function that takes the form $y = 10L^{\frac{1}{2}}K^{\frac{1}{2}}$, and assume that capital (K) is constant at $K_0 = 64$.

a. Find the marginal product of labor, $\frac{\partial y}{\partial L}$.

b. If the labor were paid real wage equivalent to the marginal product of labor, how many labors would be employed when the going wage rate is 10?

c. What happens to the number of labor demanded when the wage declines to 8?

d. How many labor would be demanded if wage remains 8, but the capital is increased to 100?

e. Find the cross partial derivative, $\frac{\partial^2 y}{\partial K \partial L}$.

8. Example: Two different demand functions are given:

$Q_1 = 11 - 2p_1 - 2p_2$ and $Q_2 = 16 - 2p_1 - 3p_2$

$TC = 10 + 4x + 2y$. Determine the quantities that maximize the profit of monopolist and also find maximum profit.

9. Two different demand functions of discriminating monopoly are given:

$p_1 = 140 - 7q_1$ and $p_2 = 90 - 0.4\, q_2/2$

$TC = 20 + 2q + 3q^2$ where $q = q_1 + q_2$, what price will the firm charge in two markets to maximize profit?

Solution:

1. a. $fxx = \frac{-4\,(5y-8)}{(2x+5y)^3}$; $fyy = \frac{-50(x+4)}{(2x+5y)^3}$ and $fxy = fyx = \frac{10x-25y+80}{(2x+5y)^3}$

b. $fxx = 294(7x + 3y)$; $fyy = 54(7x + 3y)$ and $fxy = fyx = 126\,(7x + 3y)$

c. $fxx = 30x((x^3+5y)^4 + 6x^2(x^3+5y)^3)$; $fyy = 50(x^3+5y)^3$ and $fxy = 300\,(x^3+5y)^3x^2$

d. $fxy = 2e^y$ $fxx = 0$.

e. $fxx = \frac{2-2x^2}{(1+x^2)^2}$; $fyy = 2$ and $fxy = \frac{1+2x}{1+x^2} = fyx$

f. $fxx = 6x\,e^{2y}$; $fyy = 4x^3e^{2y}$ and $fxy = 6x^2e^{2y}$

2. Derivatives
   a. $f_1 = 6x_1x_2 + 24x_1 + 5x_2 + 20$; $f_2 = 3x_1^2 + 5x_1 + 1$
   b. $f_{11} = 6x_2 + 24$; $f_{22} = 0$; $f_{12} = 6x_1 + 5$; and $f_{21} = 6x_1 + 5$. Note both cross partial derivatives are equal, as they should be, according to the Young's Theorem.
3. Answers:
   a. $\frac{\partial Q_d}{\partial Y} = 0.05$; $\frac{\partial Q_d}{\partial P_c} = 10$; $\frac{\partial Q_d}{\partial P_s} = -10P_s$
   b. 1.12
   c. - 0.56
   d. 0.16
4. Apply Young's theorm
5. Since $\frac{\partial x1}{\partial p2}$ and $\frac{\partial x2}{\partial p1}$ are both greater than zero. Hence, the commodities X_1 and X_2 are competitive.
6. $p_1 = 3.2$ and $p_2 = 3.9$; $e_{11} = -1.7$; $e_{21} = 0.8$; $e_{22} = -.2$ and $e_{12} = 0.5$.
7. Answers:

a. $\frac{\partial y}{\partial L} = 5\left(\frac{K}{L}\right)^{\frac{1}{2}} = \frac{40}{L^{\frac{1}{2}}}$

b. L = 16

c. L = 25; labor demand increases with wage decline.

d. $\frac{\partial y}{\partial L} = 5\left(\frac{K}{L}\right)^{\frac{1}{2}} = \frac{50}{L^{\frac{1}{2}}} = 8$; i.e., L = 39.

e. $2.5L^{-\frac{1}{2}}K^{-\frac{1}{2}}$

8. x=1 and y=2; $\pi = 8$

9. $p_1 = 110.52$ and $p_2 = 85.52$ and q= 13.17

6.References:

1. K. Sydaster and P. Hammond, Mathematics for Economic Analysis, Person Educational Asia, Delhi, 2002.
2. M. Hoy et.al, Mathematics for Economics, PHI Learning Private Limited, Delhi, Second Edition, 2001.
3. J.E. Draper and J.S. Klingman, Mathematical Analysis Bussiness and Economic Applications, Harper & Row Publishers, New York, 1967.
4. Rosser, Mike, Basic Mathematics for Economists Second Edition, London, 2003.

the position function $s(t)$. Thus, the acceleration may be thought of as the second derivative of position; that is,

$$a(t) = \frac{d^2s}{dt^2}$$

This notation is used in the following example.

Explore!

Change t to x in $s(t)$, $v(t)$, and $a(t)$ in Example 6.5. Use a graphing utility to graph $v(x)$ and $a(x)$ on the same coordinate axes using a viewing rectangle of [0, 2].1 by [−5, 5].5. Explain what is happening to $v(x)$ when $a(x)$ is zero. Then use your calculator to see what effect changing $s(t)$ to $s_1(t) = 2t^3 - 3t^2 + 4t$ has on $v(t)$ and $a(t)$.

EXAMPLE 6.5

If the position of an object moving along a straight line is given by $s(t) = t^3 - 3t^2 + 4t$ at time t, find its velocity and acceleration.

Solution

The velocity of the object is

$$v(t) = \frac{ds}{dt} = 3t^2 - 6t + 4$$

and its acceleration is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 6t - 6$$

HIGHER-ORDER DERIVATIVES

If you differentiate the second derivative $f''(x)$ of a function $f(x)$ one more time, you get the third derivative $f'''(x)$. Differentiate again and you get the fourth derivative, which is denoted by $f^{(4)}(x)$ since the prime notation $f''''(x)$ begins to get cumbersome. In general, the derivative obtained from $f(x)$ after n successive differentiations is called the ***n*th derivative** or **derivative of order *n*** and is denoted by $f^{(n)}(x)$.

The *n*th Derivative ▪ For any positive integer n, the nth derivative of a function is obtained from the function by differentiating successively n times. If the original function is $y = f(x)$, the nth derivative is denoted by

$$\frac{d^ny}{dx^n} \quad \text{or} \quad f^{(n)}(x)$$

EXAMPLE 6.6

Find the fifth derivative of each of the following functions:

(a) $f(x) = 4x^3 + 5x^2 + 6x - 1$

(b) $y = \dfrac{1}{x}$

Solution

(a) $f'(x) = 12x^2 + 10x + 6$

$f''(x) = 24x + 10$

$f'''(x) = 24$

$f^{(4)}(x) = 0$

$f^{(5)}(x) = 0$

(b) $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^{-1}) = -x^{-2} = -\dfrac{1}{x^2}$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(-x^{-2}) = 2x^{-3} = \dfrac{2}{x^3}$

$\dfrac{d^3y}{dx^3} = \dfrac{d}{dx}(2x^{-3}) = -6x^{-4} = -\dfrac{6}{x^4}$

$\dfrac{d^4y}{dx^4} = \dfrac{d}{dx}(-6x^{-4}) = 24x^{-5} = \dfrac{24}{x^5}$

$\dfrac{d^5y}{dx^5} = \dfrac{d}{dx}(24x^{-5}) = -120x^{-6} = -\dfrac{120}{x^6}$

P . R . O . B . L . E . M . S 2.6

In Problems 1 through 16, find the second derivative of the given function. In each case, use the appropriate notation for the second derivative and simplify your answer. (Don't forget to simplify the first derivative as much as possible before computing the second derivative.)

1. $f(x) = 5x^{10} - 6x^5 - 27x + 4$

2. $f(x) = \dfrac{2}{5}x^5 - 4x^3 + 9x^2 - 6x - 2$

3. $y = 5\sqrt{x} + \dfrac{3}{x^2} + \dfrac{1}{3\sqrt{x}} + \dfrac{1}{2}$

4. $y = \dfrac{2}{3x} - \sqrt{2x} + \sqrt{2}\,x - \dfrac{1}{6\sqrt{x}}$

5. $f(x) = (3x + 1)^5$

6. $f(t) = \dfrac{2}{5t + 1}$

7. $h = (t^2 + 5)^8$

8. $y = (1 - 2x^3)^4$

9. $f(x) = \sqrt{1 + x^2}$

10. $f(u) = \dfrac{1}{(3u^2 - 1)^2}$

11. $z = \frac{2}{1 + x^2}$

12. $y = \frac{t}{(t + 1)^2}$

13. $f(x) = x(2x + 1)^4$ (Use the product rule.)

14. $f(x) = 2x(x + 4)^3$ (Use the product rule.)

15. $y = \left(\frac{t}{t + 1}\right)^2$

16. $h = \frac{(x - 2)^3}{x^2}$

In Problems 17 through 20, the position s(t) *of an object moving along a straight line is given. In each case,*

(a) Find the object's velocity v(t) *and acceleration* a(t).

(b) Find all times t *where the acceleration is 0.*

17. $s(t) = 3t^5 - 5t^3 - 7$

18. $s(t) = 2t^4 - 5t^3 + t - 3$

19. $s(t) = (1 - t)^3 + (2t + 1)^2$

20. $s(t) = 4t^{5/2} - 15t^2 + t - 3$

WORKER EFFICIENCY 21. An efficiency study of the morning shift at a certain factory indicates that an average worker arriving on the job at 8:00 A.M. will have produced $Q(t) = -t^3 + 8t^2 + 15t$ units t hours later.

(a) Compute the worker's rate of production at 9:00 A.M.

(b) At what rate is the worker's rate of production changing with respect to time at 9:00 A.M.?

(c) Use calculus to estimate the change in the worker's rate of production between 9:00 and 9:15 A.M.

(d) Compute the actual change in the worker's rate of production between 9:00 and 9:15 A.M.

INFLATION 22. It is projected that t months from now, the average price per unit for goods in a certain sector of the economy will be $P(t) = -t^3 + 7t^2 + 200t + 300$ dollars.

(a) At what rate will the price per unit be increasing with respect to time 5 months from now?

(b) At what rate will the rate of price increase be changing with respect to time 5 months from now?

(c) Use calculus to estimate the change in the rate of price increase during the first half of the sixth month.

(d) Compute the actual change in the rate of price increase during the first half of the sixth month.

POPULATION GROWTH 23. Suppose that a 5-year projection of population trends suggests that t years from now, the population of a certain community will be $P(t) = -t^3 + 9t^2 + 48t + 200$ thousand.

(a) At what rate will the population be growing 3 years from now?

(b) At what rate will the rate of population growth be changing with respect to time 3 years from now?

(c) Use calculus to estimate the change in the rate of population growth during the first month of the fourth year.
(d) Compute the actual change in the rate of population growth during the first month of the fourth year.

ACCELERATION 24. An object moves along a straight line so that after t seconds, its distance from its starting point is $D(t) = t^3 - 12t^2 + 100t + 12$ meters. Find the acceleration of the object after 3 seconds.

ACCELERATION 25. After t hours of an 8-hour trip, a car has gone $D(t) = 64t + \frac{10}{3}t^2 - \frac{2}{9}t^3$ kilometers.
(a) Derive a formula expressing the acceleration of the car as a function of time.
(b) At what rate is the velocity of the car changing with respect to time at the end of 6 hours? Is the velocity increasing or decreasing at this time?
(c) By how much does the velocity of the car actually change during the seventh hour?

MEDICINE 26. One biological model* suggests that the human body's reaction to a dose of medicine can be modeled by a function of the form

$$F = \frac{1}{3}(KM^2 - M^3)$$

where K is a positive constant and M is the amount of medicine absorbed in the blood. The derivative $S = \frac{dF}{dM}$ can be thought of as a measure of the sensitivity of the body to the medicine.
(a) Find the sensitivity S.
(b) Find $\frac{dS}{dM} = \frac{d^2F}{dM^2}$ and give an interpretation of the second derivative.

BLOOD PRODUCTION 27. A model† for the production of a type of white blood cells (granulocytes) uses a function of the form

$$p(x) = \frac{Ax}{B + x^m}$$

where A and B are positive constants, the exponent m is positive, and x is the number of cells present.
(a) Find the rate of production $p'(x)$.
(b) Find $p''(x)$ and determine all values of x for which $p''(x) = 0$ (your answer will involve m).

* Thrall et al., *Some Mathematical Models in Biology*, U.S. Dept. of Commerce.
† M. C. Mackey and L. Glass, "Oscillations and Chaos in Physiological Control Systems," *Science*, Vol. 197, pages 287–289.

(c) Read an article on blood cell production and write a paragraph on how mathematical methods can be used to model such production.‡

ACCELERATION 28. If an object is dropped or thrown vertically, its height (in feet) after t seconds is $H(t) = -16t^2 + S_0t + H_0$, where S_0 is the initial speed of the object and H_0 its initial height.

(a) Derive an expression for the acceleration of the object.

(b) How does the acceleration vary with time?

(c) What is the significance of the fact that the answer to part (a) is negative?

29. Find $f^{(4)}(x)$ if $f(x) = x^5 - 2x^4 + x^3 - 3x^2 + 5x - 6$.

30. Find $\dfrac{d^3y}{dx^3}$ if $y = \sqrt{x} - \dfrac{1}{2x} + \dfrac{x}{\sqrt{2}}$.

31. Find $f'''(x)$ if $f(x) = \dfrac{1}{\sqrt{3x}} - \dfrac{2}{x^2} + \sqrt{2}$.

32. Find $xf''(x) - 2f'(x) - \dfrac{4}{x}f(x)$ if $f(x) = x^3 - \sqrt{x} + \dfrac{1}{x}$.

33. An object moves along a straight line with velocity $v(t) = (7t - 5)^2(4 - t)^3$ for $0 \le t \le 5$. Use your graphing utility to draw the graphs of the velocity and the acceleration $a(t)$ on the same axes and then answer these questions.

(a) When is the object accelerating and when is it decelerating for $0 \le t \le 5$?

(b) When is the velocity the largest for $0 \le t \le 5$?

(c) When is the acceleration the largest for $0 < t < 5$? What is the velocity at this time?

(d) What is the difference between the largest and smallest acceleration for $0 \le t \le 5$?

34. An object moves along a straight line in such a way that its position at time t is given by $s(t) = (2 + t - t^2)^{3/2}$. Use a graphing utility to sketch the graph of $s(t)$, the velocity $v(t)$, and the acceleration $a(t)$ on the same axes for $0 \le t < 2$. Then use your calculator to answer the following questions.

(a) When is the velocity 0 for $0 \le t < 2$? Where is the object and what is its acceleration at this time?

(b) When is the acceleration 0 for $0 \le t < 2$? Where is the object and what is its velocity at this time?

(c) Does the acceleration have a maximum value for $0 \le t < 2$? If so, what is the maximum acceleration?

‡You may wish to begin your investigation with the article by William B. Gearhart and Mario Martelli, "Blood Cell Population Model, Dynamical Diseases, and Chaos," *UMAP Module 1990*, Arlington, MA Consortium for Mathematics and Its Applications, Inc., 1991.

10. The Second-Order Derivative, Concavity and Convexity

The second-order derivative (or simply 'second derivative') is the derivative of the derivative

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h}.$$

The objective of this section is to provide examples of their application in general, and in economics in particular.

We can also compute 'third-order derivatives', 'fourth-order derivatives', etc. These are useful, although we will be using them less frequently.

Example 10.1 If $f(x) = x^3$, then $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$.

Notation If $y = f(x)$, then

Second Derivative $f''(x)$, $f^{(2)}(x)$, y'', $\frac{d^2}{dx^2}f(x)$, $\frac{d^2y}{dx^2}$

n-th ordered derivative $f^{(n)}(x)$, $y^{(n)}$, $\frac{d^n}{dx^n}f(x)$, $\frac{d^ny}{dx^n}$

If y is a function of time, then the second derivative is often written $\ddot{y}$, following the convention that the first-derivative in such applications is written $\dot{y}$.

The most straightforward interpretation of the second-derivative is that it is the rate of change of the rate of change of $f(x)$ as x increases.

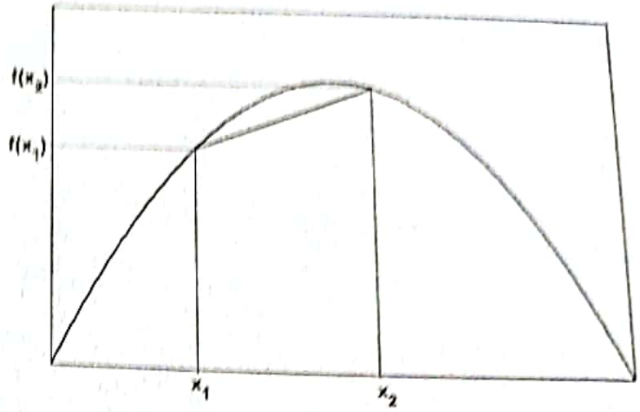
Geometrically, the second derivative can also help to describe the shape of a function. When used in specific disciplines, the first and second derivations can often be given an interpretation appropriate to the context of that discipline. You are probably familiar with the concepts of velocity and acceleration in physics.

Example 10.2 If $x(t)$ gives you the position of an object relative to some reference point $x(0)$, and t is time, then $\dot{x}$ is the velocity of the object, and $\ddot{x}$ is its acceleration.

Geometric Interpretations

We can use the second-derivative to characterize the shape of a function, particularly regarding its curvature. We begin with a geometric definition of the concept of convexity and concavity.

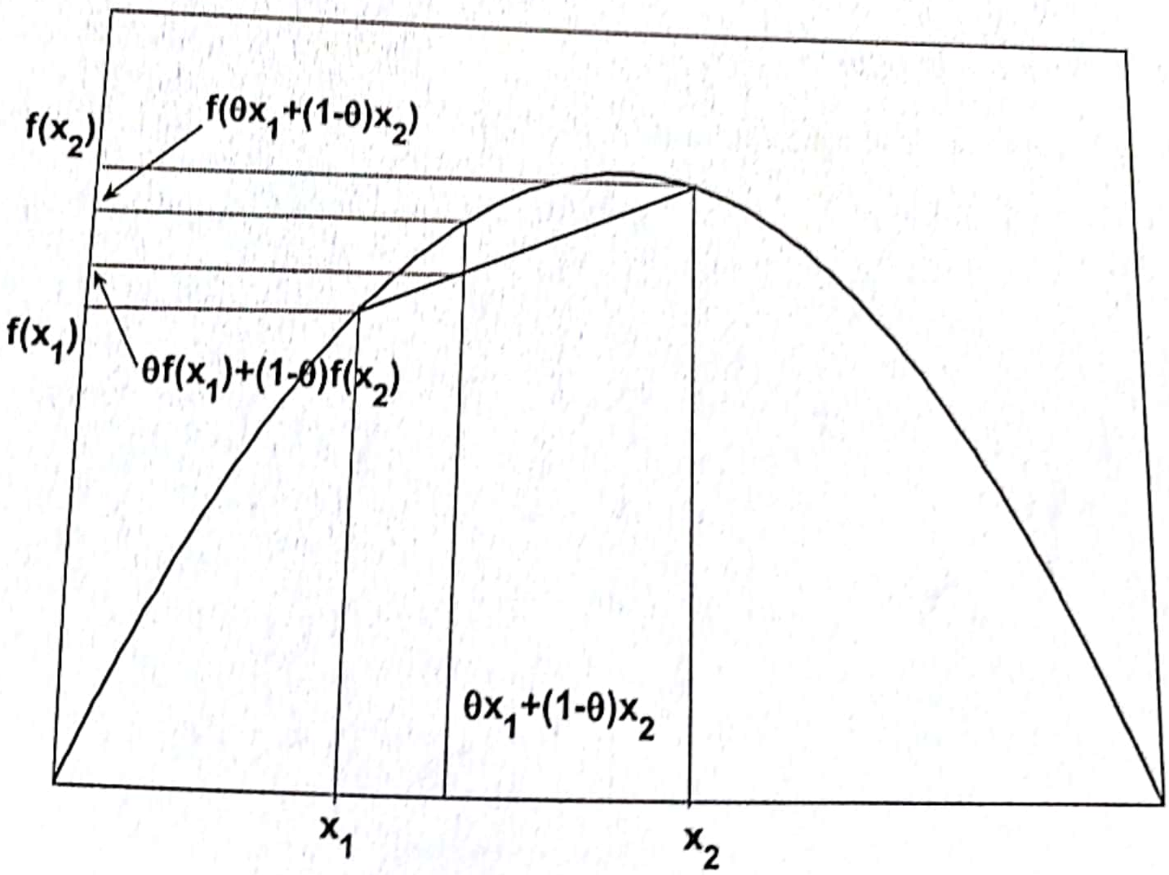
A function is said to be strictly concave if for any $x_1 < x_2$, the chord joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies strictly below the function for all values of x strictly between x_1 and x_2, $x_1 < x < x_2$



One way to express this mathematically is given in next definition.

Definition A function is **strictly concave** over its domain if for any two values of x, say x_1 and x_2 with $x_1 < x_2$, and for all $0 < \theta < 1$,

$$\theta f(x_1) + (1-\theta) f(x_2) < f(\theta x_1 + (1-\theta) x_2)$$



If we allow the chord to lie on the function at some points, then the function is said to be weakly concave, or simply 'concave'. A function is **concave** if for any two values of x, say x_1 and x_2 with $x_1 < x_2$, and for all $0 \le \theta \le 1$, we have

$$\theta f(x_1) + (1-\theta) f(x_2) \le f(\theta x_1 + (1-\theta)x_2).$$

A function is **strictly convex** if for any two values of x, say x_1 and x_2 with $x_1 < x_2$, and for all $0 < \theta < 1$, we have

$$\theta f(x_1) + (1-\theta) f(x_2) > f(\theta x_1 + (1-\theta)x_2).$$

A function is **convex** if for any two values of x, say x_1 and x_2 with $x_1 < x_2$, and for all $0 < \theta < 1$, we have

$$\theta f(x_1) + (1-\theta) f(x_2) \ge f(\theta x_1 + (1-\theta)x_2),\ 0 \le \theta \le 1.$$

Note that strictly concave functions are concave; the set of strictly concave functions are a subset of the set of concave functions. Likewise, strictly convex functions are convex. Note also that strictly concave functions can be strictly increasing or strictly decreasing.

Example 10.3 $f(x) = x^{1/2}$ is strictly concave, and strictly increasing.

The definitions of concavity given above are very general, and can apply to functions that may be non-differentiable at certain points. For differentiable functions, however, it is often easier to make use of the second derivations to show concavity/convexity. From these definitions given earlier, it is possible to show that

(1) f is convex on an interval I $\Leftrightarrow$ $f''(x) \ge 0$ for all x in I

(2) f is concave on an interval I $\Leftrightarrow$ $f''(x) \le 0$ for all x in I

(3) $f''(x) > 0$ for all x in I $\Rightarrow$ f is strictly convex on an interval I

(4) $f''(x) < 0$ for all x in I $\Rightarrow$ f is strictly concave on an interval I

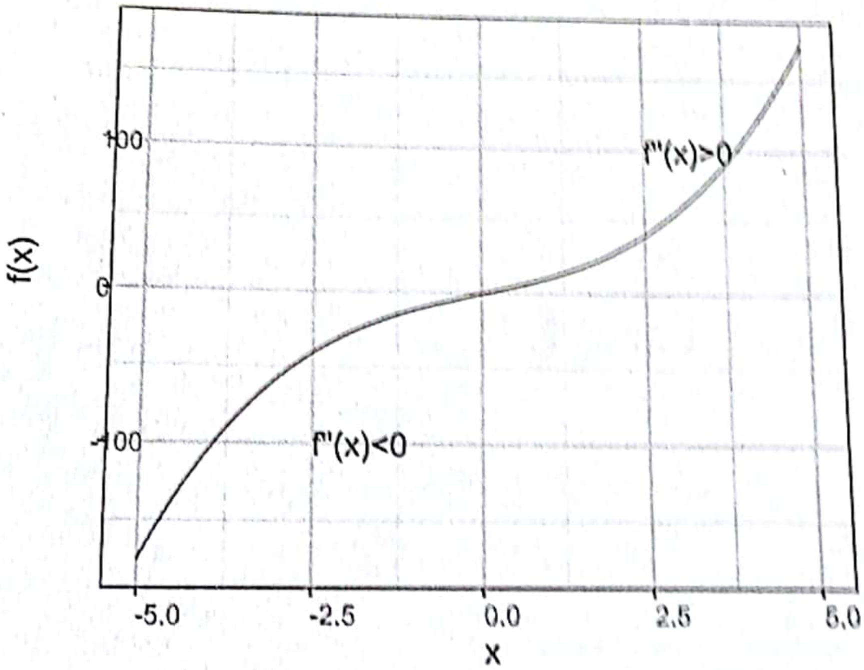
We omit proofs. The results are sufficiently intuitive for our purpose. For a function to be concave, it is apparent that the slope must never increase as x increases. It is also apparent that if the slope of the function always decreases as x increases, i.e., if $f''(x)<0$, then the function will have a strictly concave shape.

The only aspect of definitions (1) to (4) that may not be obvious is that it is possible for a function to be strictly concave over an interval I without $f''(x)<0$ for all x in I.

Example 10.4 The function $f(x)=-x^4$ is strictly concave, but it is not true that $f''(x)<0$ for all x. In particular, $f''(x)=-12x^2=0$ at $x=0$.

Convexity and concavity will play an important role in function optimization. For instance, if a strictly concave function $f(x)$ has a point x^* at which $f(x^*)=0$, then x^* must be an optimal point.

A function may be concave in some portions of its domain, and convex in other parts of it. The point where a function switches from concavity to convexity, or the other way around, is called an **inflection point**. The plot below is of a function $f(x)=x^3+10x$, its first derivative is $f'(x)=3x^2+10$ and its second derivative is $f''(x)=6x$. It is concave when $x<0$ and convex when $x>0$.



Definition The point x_0 is called an **inflection point** of a function $f(x)$ if over some interval (a,b) containing x_0, we have

$$f''(x)\leq x \text{ over } (a,x_0) \text{ and } f''(x)\geq 0 \text{ over } (x_0,b)$$

or

$$f''(x)\geq x \text{ over } (a,x_0) \text{ and } f''(x)\leq 0 \text{ over } (x_0,b)$$

If x_0 is an inflection point, it must be that $f''(x_0)=0$. But note that $f''(x_0)=0$ does not imply that x_0 is an inflection point. (example: $f(x)=x^4$). However, if $f''(x_0)=0$ and $f''(x_0)$ changes sign at x_0, then x_0 is an inflection point. Note also that $f'(x_0)$ need not be zero for x_0 to be an inflection point, as in the function plotted above.

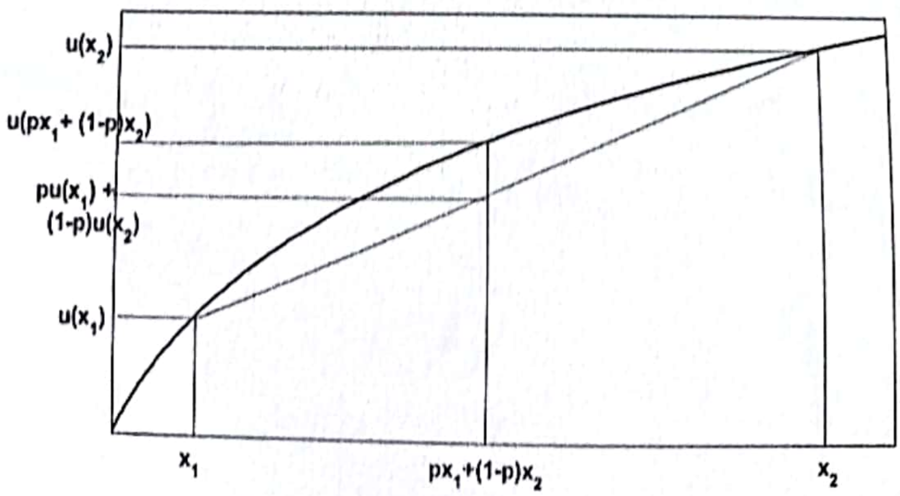
Some Economic Applications

The second derivative is useful in economics not only because of its role in optimization, but also because it can often be given economic interpretations.

<u>Example 10.5</u> *Diminishing Marginal Productivity* As a firm increases labor input starting from low levels, its production capacity generally increases. However, the increase in productive capacity, as more labor is added, tends to get smaller: increasing labor from 100 to 101 might increase productive capacity by 10 units; increasing labor from 101 to 102 might increase productive capacity by 7 units, say. That is, the rate of increase in productive capacity falls as more labor is added. Suppose we represent the firm's production function by $f(L)$ where L is labor input. We can impose increasing productivity be requiring the condition $f'(L)>0$. We can impose "diminishing marginal productivity" by requiring $f''(x)<0$.

<u>Example 10.6</u> *Diminishing Marginal Utility* Suppose $u(x)$ represents the utility that a person gets from consuming x amount of a good. If $u'(x)>0$ then more of the good is preferred to less of it. If $u''(x)<0$, then every additional unit of the good generates less additional utility for the consumer than the previous additional unit.

<u>Example 10.7</u> *Risk aversion* Suppose again that $u'(x)>0$ and $u''(x)<0$. Such a function has a shape like in the figure below.



Suppose the person with utility function $u(x)$ is given a choice. He can either accept a lottery that pays x_1 with probability p, and x_2 with probability $1-p$, where $0<p<1$, or accept $px_1+(1-p)x_2$ with certainty. In the first case, the person's expected utility is $pu(x_1)+(1-p)u(x_2)$. In the second case, the person gets utility $u(px_1+(1-p)x_2)$.

In the figure we see that because of the shape of the function,

$$pu(x_1)+(1-p)u(x_2) < u(px_1+(1-p)x_2)$$

This person would rather have the certain amount. He would avoid the lottery if given a choice. He is 'risk-averse'. Contrast this with the fellow for whom

$$pu(x_1)+(1-p)u(x_2) > u(px_1+(1-p)x_2).$$

The person prefers the lottery. He is 'risk-loving'.

Exercise

1. Show that the function $f(x)=e^{x-1}-x$ is strictly convex.
2. Find the regions of x for which $f(x)=(x-1)(x-2)(x+3)$ is (i) strictly convex, (ii) strictly concave.
3. Find the first and second derivatives of each of the following functions.

   (a) $f(\mu) = \sum_{i=1}^{n}(x_i-\mu)^2$ where x_i, $i=1,2,\ldots,n$ is a set of numbers. Find the intervals over which the function is (strictly) increasing, decreasing, concave, or convex.

   (b) $y = f(x) = \log_x e$. What is the largest possible domain of this function? Find the intervals over which the functions are (strictly) increasing, decreasing, concave, or convex. Sketch the graph

LOCAL EXTREMA AND POINTS OF INFLECTION

In lecture 9, we have seen a necessary condition for local maximum and local minimum. In this lecture we will see some sufficient conditions.

In the following results we assume $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$.

1. Sufficient conditions for a local extremum

We will state results for local maximum, and results for local minimum are similar.

Theorem 1. *Let f be continuous at c. If for some $\delta > 0$, f is increasing on $(c - \delta, c)$ and decreasing on $(c, c + \delta)$, then f has a local maximum at c.*

Proof. Choose x and y such that $c - \delta < x < y < c$. Then $f(x) \leq f(y)$. The continuity of f at c implies that $f(x) \leq \lim_{y \to c^-} f(y) = f(c)$. Similarly, if $c < y < x < c + \delta$, then $f(x) \leq \lim_{y \to c^+} f(y) = f(c)$. This proves the result. □

Corollary 2. *(1) **(First Derivative Test for Local Maximum)** Let f be continuous at c. If for some $\delta > 0$*

$$f'(x) \geq 0 \ \forall\, x \in (c - \delta, c) \text{ and } f'(x) \leq 0 \ \forall\, x \in (c, c + \delta),$$

then f has a local maximum at c.

*(2) **(Second Derivative Test for Local Maximum)** If f is twice differentiable at c and satisfies $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

Remark 3. *An easy way to remember the First Derivative Test (for local minimum and local maximum) is as follows:*

f' changes from $-$ to $+$ at $c \Rightarrow f$ has a local minimum at c,

f' changes from $+$ to $-$ at $c \Rightarrow f$ has a local maximum at c.

Example 4. *(1) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \frac{1}{x^4 - 2x^2 + 7}$. We have $f'(x) = \frac{-4x(x-1)(x+1)}{(x^4-2x^2+7)^2}$. Thus, $f'(x) = 0$ when $x = -1, 0, 1$. Now, consider the following table,*

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Sign of f'	$+$	$-$	$+$	$-$

So, we conclude that f has a local minimum at $x = 0$ and a local maxima at

(2) *Consider $f : (-1,1) \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} x^2 & \text{if } 0 < |x| < 1, \\ -1, & \text{if } x = 0. \end{cases}$$

We note the conditions of the first derivative test is not satisfied. In fact, f is differentiable on $(-1,0)$ and $(0,1)$ and f' changes sign from $-$ to $+$ at $x = 0$ but f is not continuous at $x = 0$. Nevertheless, $f(0) < f(x)$ for all nonzero $x \in (-1,1)$, and thus f has a strict local minimum at $x = 0$.

(3) *If $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^4$. Then $f(0) = 0 < f(x)$ for all nonzero $x \in \mathbb{R}$). Therefore, f has a strict local minimum at $x = 0$. Note that $f'(0) = 0$, but $f''(0)$ is not positive.*

2. Convex Sets and Convex Functions

Let V be a vector space over $\mathbb{R}$.

Definition 5. *A set $C \subseteq V$ is said to be convex if the line segment between any two points in C lies in C, i.e.,*

if for any $x, y \in C$ and any $t \in [0,1]$, we have $tx + (1-t)y \in C$.

Example 6. *It is clear that the unit disc is convex in $\mathbb{R}^2$. However, the unit circle is not convex. Any interval in $\mathbb{R}$ is a convex set.*

Definition 7. *Let $C \subseteq V$ be a convex set. A function $f : C \to \mathbb{R}$ is said to be* **convex** *if for all $x, y \in C$ and for all $t \in [0,1]$, we have*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y). \tag{$\star$}$$

If for $t \in (0,1)$, the above inequality is strict, the f is said strictly convex.

We say that f is **concave** *if the reverse inequality in $(\star)$ holds.*

Theorem 8 (Derivative Test for Convexity). *Assume that $f : [a,b]$ is differentiable on (a,b). If f' is increasing on (a,b), then f is convex on $[a,b]$. In particular, if f'' exists and non-negative on (a,b), then f is convex.*

Example 9. *Let $f(x) = x^3 - 6x^2 + 9x$. We have $f'(x) = 3(x-1)(x-3)$ and $f''(x) = 6x - 12$. We see that $f''(x) > 0$ if $x > 2$ and $f''(x) < 0$ if $x < 2$. Hence, f is convex for $x > 2$ and concave for $x < 2$.*

Example 10 (Examples of convex functions).

- *$x \log x$ is strictly convex on $(0, \infty)$.*
- *$f(x) = x^4$ is strictly convex but $f''(0) = 0$*
- *e^x is strictly convex on $\mathbb{R}$.*

The following result is one of the reasons why convex functions are very useful in applications especially in optimization problems.

Theorem 11. *If $f : (a, b) \to$ is convex and $c \in (a, b)$ is a local minimum, then is c a minimum for f on (a, b). That is, local minima of convex functions are global minima.*

3. Points of Inflection

Definition 12. *Let $f : (a, b) \to \mathbb{R}$ be a function and $c \in (a, b)$. The point c is said to be a* **point of inflection** *for f if there is $\delta > 0$ such that f is convex in $(c - \delta, c)$, while f is concave in $(c, c + \delta)$, or vice versa, that is, f is concave in $(c - \delta, c)$, while f is convex in $(c, c + \delta)$.*

Example 13. *For the function $f(x) = x^3$ on $\mathbb{R}$, 0 is a point of inflection.*

Theorem 14. *Let $f : (a, b) \to \mathbb{R}$ be a function and $c \in (a, b)$.*

*(1) **Necessary Condition for a Point of Inflection]** Let f be twice differentiable at c. If c is a point of inflection for f, then $f''(c) = 0$.*

*(2) **[Sufficient Condition for a Point of Inflection]** Let f be thrice differentiable at c. If $f''c) = 0$ and $f'''(c) = 0$, then c is a point of inflection for f.*

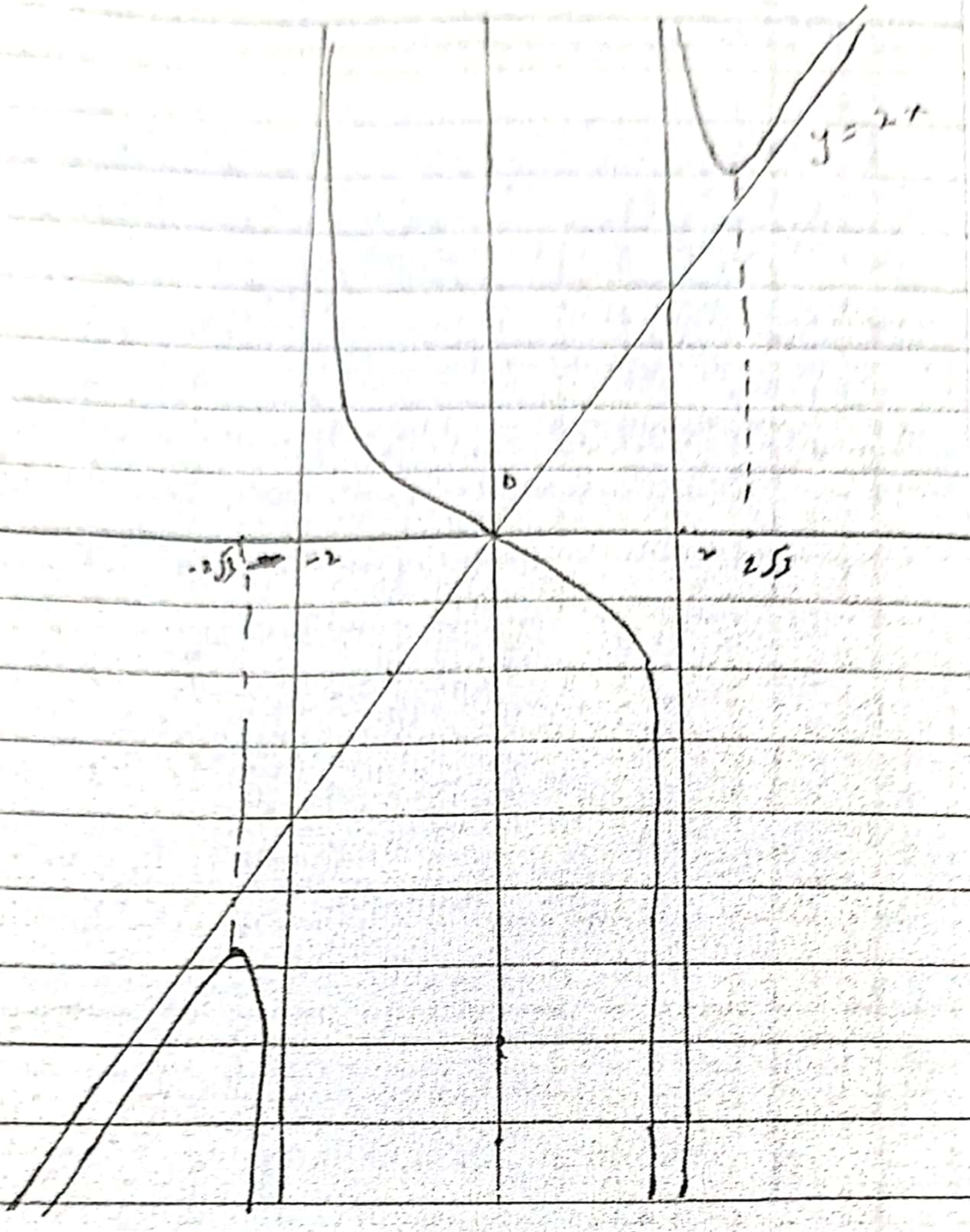
Example 15.
- *For the function $f(x) = x^4$, 0 is not a point of inflection, though, $f''(0) = 0$.*
- *For the function $f(x) = x^5$, 0 is a point of inflection, but $f'''(0) = 0$.*

Problem 16. *Sketch the graph of the function $f(x) = \frac{2x^3}{x^2-4}$ after finding the intervals of decrease/increase, intervals of concavity/convexity, points of local minima/local maxima, points of inflection and asymptotes.*

Solution. We note that

$$f(x) = 2x + \frac{8x}{x^2 - 4}, \; f'(x) = \frac{2x^2(x^2 - 12)}{(x^2 - 4)^2} \text{ and } f''(x) = \frac{16x(x^2 + 12)}{(x^2 - 4)^3}.$$

Verify that $x = 2$, $x = -2$ and $y = 2x$ are the asymptotes. Moreover, the function is increasing on $(-\infty, -2\sqrt{3})$ and $(2\sqrt{3}, \infty)$. The function is decreasing on $(-2\sqrt{3}, -2)$, $(-2, 2)$ and $(2, 2\sqrt{3})$. Furthermore, the function is convex on $(-2, 0)$ and $(2, \infty)$ and concave on $(-\infty, -2)$ and $(0, 2)$. The point of inflection is 0. The sketch of the graph is shown below.


0
-2√3
-2
2
2√3

Solution
Compute the first derivative

$$f'(x) = 20x^3 - 6x - 3$$

and then differentiate again to get

$$f''(x) = 60x^2 - 6$$

EXAMPLE 6.2

Find the second derivative of the function $y = (x^2 + 1)^5$.

Solution
Compute the first derivative, using the general power rule, to get

$$\frac{dy}{dx} = 5(x^2 + 1)^4(2x) = 10x(x^2 + 1)^4$$

Then differentiate again, using the product rule, to get

$$\begin{aligned}\frac{d^2y}{dx^2} &= 10x[4(x^2 + 1)^3(2x)] + 10(x^2 + 1)^4 \\ &= 80x^2(x^2 + 1)^3 + 10(x^2 + 1)^4 \\ &= 10(x^2 + 1)^3[8x^2 + (x^2 + 1)] \\ &= 10(x^2 + 1)^3(9x^2 + 1)\end{aligned}$$

EXAMPLE 6.3

Find the second derivative of the function $f(x) = \dfrac{3x - 2}{(x - 1)^2}$.

Solution
By the quotient rule,

$$\begin{aligned}f'(x) &= \frac{(x - 1)^2(3) - (3x - 2)[2(x - 1)(1)]}{(x - 1)^4} \\ &= \frac{(x - 1)[3(x - 1) - 2(3x - 2)]}{(x - 1)^4} \\ &= \frac{3x - 3 - 6x + 4}{(x - 1)^3} \\ &= \frac{1 - 3x}{(x - 1)^3}\end{aligned}$$

The Second Derivative

This section is about the rate of change of the rate of change of a quantity. Such rates arise in a variety of situations. For example, the acceleration of a car is the rate of change with respect to time of its velocity, which is itself the rate of change with respect to time of its position. If position is measured in miles and time in hours, the velocity (or rate of change of distance) is measured in miles per hour, and the acceleration (or rate of change of velocity) is measured in miles per hour per hour.

Statements about the rate of change of a rate of change are used frequently in economics. In inflationary times, for example, you may hear a government economist assure the nation that although the inflation rate is increasing, the rate at which it is doing so is decreasing. That is, prices are still going up, but not as quickly as they were before.

The rate of change of the function $f(x)$ with respect to x is the derivative $f'(x)$, and likewise, the rate of change of the function $f'(x)$ with respect to x is *its* derivative $(f'(x))'$. This notation is awkward, so we write the derivative of the derivative of $f(x)$ as $(f'(x))' = f''(x)$ and refer to it as the *second derivative* of $f(x)$ (read $f''(x)$ as "f double prime of x"). If $y = f(x)$, then the second derivative of y with respect to x is written as y'' or as $\frac{d^2y}{dx^2}$. Here is a summary of the terminology and notation used for second derivatives.

Note

The ordinary derivative $f'(x)$ is sometimes called the **first derivative** to distinguish it from the **second derivative** $f''(x)$.

The Second Derivative ■ The second derivative of a function is the derivative of its derivative. If $y = f(x)$, the second derivative is denoted by

$$\frac{d^2y}{dx^2} \quad \text{or} \quad f''(x)$$

The second derivative gives the rate of change of the rate of change of the original function.

ATION OF THE DERIVATIVE

You don't have to use any new rules to find the second derivative of a function. Just find the first derivative and then differentiate again.

EXAMPLE 6.1

Find the second derivative of the function $f(x) = 5x^4 - 3x^2 - 3x + 7$.

By the quotient rule again,

$$\begin{aligned} f''(x) &= \frac{(x-1)^3(-3) - (1-3x)[3(x-1)^2(1)]}{(x-1)^6} \\ &= \frac{-3(x-1)^2[(x-1) + (1-3x)]}{(x-1)^6} \\ &= \frac{-3(-2x)}{(x-1)^4} = \frac{6x}{(x-1)^4} \end{aligned}$$

A Word of Advice

Before computing the second derivative of a function, always take the time to simplify the first derivative as much as possible. The more complicated the form the first derivative is, the more tedious the computation of the second derivative will be.

Applications of the Second Derivative

The second derivative will be used in Chapter 3, Section 2, to obtain information about the shapes of graphs. In Sections 4 and 5 of that chapter, the second derivative will appear again, this time in the solution of optimization problems. Here is a more elementary application illustrating the interpretation of the second derivative as the rate of change of a rate of change.

EXAMPLE 6.4

An efficiency study of the morning shift at a certain factory indicates that an average worker who arrives on the job at 8:00 A.M. will have produced

$$Q(t) = -t^3 + 6t^2 + 24t$$

units t hours later.

(a) Compute the worker's rate of production at 11:00 A.M.

(b) At what rate is the worker's rate of production changing with respect to time at 11:00 A.M.?

(c) Use calculus to estimate the change in the worker's rate of production between 11:00 and 11:10 A.M.

(d) Compute the actual change in the worker's rate of production between 11:00 and 11:10 A.M.

Solution

(a) The worker's rate of production is the first derivative

$$Q'(t) = -3t^2 + 12t + 24$$

of the output function $Q(t)$. At 11:00 A.M., $t = 3$ and the rate of production is

$$Q'(3) = -3(3)^2 + 12(3) + 24 = 33 \text{ units per hour}$$

(b) The rate of change of the rate of production is the second derivative

$$Q''(t) = -6t + 12$$

of the output function. At 11:00 A.M., this rate is

$$Q''(3) = -6(3) + 12 = -6 \text{ units per hour per hour}$$

The minus sign indicates that the worker's rate of production is decreasing; that is, the worker is slowing down. The rate of this decrease in efficiency at 11:00 A.M. is 6 units per hour per hour.

(c) Note that 10 minutes is $\frac{1}{6}$ hour. To estimate the change in the production rate $Q'(t)$ due to a change in t of $\Delta t = \frac{1}{6}$ hour, apply the approximation formula from Section 4 to the function $Q'(t)$ to get

$$\text{Change in rate of production} = \Delta Q' \simeq Q''(t)\,\Delta t$$

Evaluate this expression when $t = 3$ and $\Delta t = \frac{1}{6}$ to conclude that

$$\begin{array}{l}\text{Change in rate}\\ \text{of production}\end{array} \simeq Q''(3)\Delta t = -6\left(\frac{1}{6}\right) = -1 \text{ unit per hour}$$

That is, the worker's rate of production (which was 33 units per hour at 11:00 A.M.) will decrease by approximately 1 unit per hour (to approximately 32 units per hour) during the subsequent 10 minutes.

(d) The actual change in the worker's rate of production between 11:00 and 11:10 A.M. is the difference between the values of the rate $Q'(t)$ when $t = 3$ and when $t = 3\frac{1}{6} = \frac{19}{6}$. That is,

$$\begin{array}{l}\text{Actual change in}\\ \text{rate of production}\end{array} = Q'\left(\frac{19}{6}\right) - Q'(3)$$

$$= \left[-3\left(\frac{19}{6}\right)^2 + 12\left(\frac{19}{6}\right) + 24\right] - [-3(3)^2 + 12(3) + 24]$$

$$\simeq 31.92 - 33 = -1.08 \text{ units per hour}$$

Thus, by 11:10 A.M., the worker's rate of production, which was 33 units per hour at 11:00 A.M., will actually have decreased by 1.08 units per hour to 31.92 units per hour.

Recall from Section 2 that the **acceleration** $a(t)$ of an object moving along a straight line is the derivative of the velocity $v(t)$, which in turn is the derivative of

15. Implicit Functions and Their Derivatives

When y is written as a function of $(x_1, \ldots, x_m)$,

$$y = f(x_1, \ldots, x_m)$$

we say that y is an *explicit function of* $(x_1, \ldots, x_m)$.

Things are different when y and $(x_1, \ldots, x_m)$ are combined in a single function so that

$$f(x_1, \ldots, x_m, y) = 0. \tag{15.0.1}$$

If the $x_1, \ldots, x_m$ determine y in equation (15.0.1), we say that y is an *implicit function of* $(x_1, \ldots, x_m)$

With luck, we will be able to solve for y in terms of $(x_1, \ldots, x_m)$. But that is not always possible. For example, the quintic equation

$$y^5 - 5xy + 4x^2 = 0$$

does not have an explicit solution, although we can say that $(x, y) = (1, 1)$ is a solution, as is $(1/4, 1)$, suggesting that $y(1) = 1$ and $y(1/4) = 1$. It's also clear that $y(0) = 0$. There are hints of a function here, but we can't solve for it.

When the equation implicitly defines y in terms of x, but we cannot write an expression for $y(x)$, we might still be able to determine the derivatives. The Implicit Function Theorem gives conditions for finding local functions for y and their derivatives.

15.1 Is there an Implicit Function?

One issue with equation (15.0.1) is that it is difficult to determine whether there even is an implicit function.

► **Example 15.1.1: No Implicit Function for a Circle.** Consider the equation $x^2 + y^2 = 25$. Does this implicitly define $y(x)$? In this case we can solve for y, obtaining

$$y(x) = \pm\sqrt{25 - x^2}.$$

There is a problem here. This is not a function!

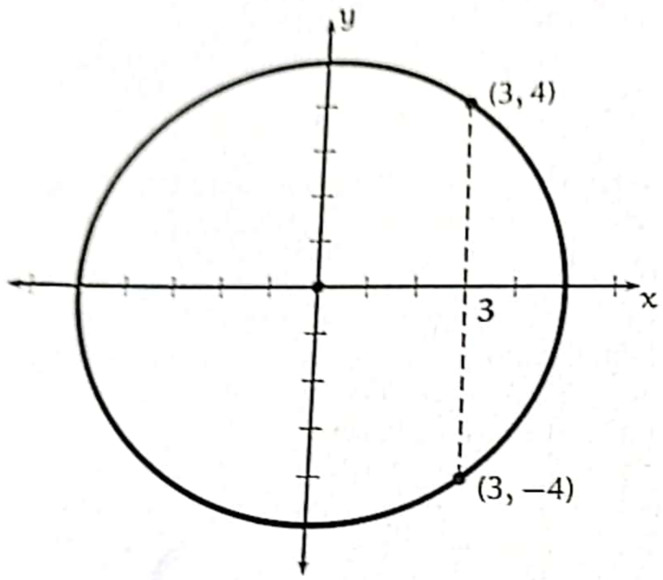


Figure 15.1.2: The circle is the graph of $x^2 + y^2 = 25$, which tries to implicitly define y as a function of x. As you can see, there are two solutions $y(x)$ for most values of x. This is illustrated at $x = 3$.

◄

For values of $x \in (-5, +5)$, there are two values of $y(x)$, not one. Only at $x = \pm 5$ do we have a function. Everywhere else there are two values of y for every x. This is illustrated in Figure 15.2.2 where there are two values of y corresponding to $x = 3$.

One way to work around this is to lower the bar, to give up the search for a global function and focus on a locally defined implicit function. We look for a function that solves the equation in a neighborhood of a point (x_0, y_0). We can use one function near $(3, 4)$ and another near $(3, -4)$.

15.2 Picking an Implicit Function

▸ Example 15.2.1: **Local Implicit Functions on a Circle.** Here $y = +(25 - x^2)^{1/2}$ is implicitly defined by the equation $x^2 + y^2 = 25$ and includes the starting point $(x_0, y_0) = (3, 4)$. It can be defined on open sets as large as $(-5, 5)$. Similarly, if $(x_0, y_0) = (3, -4)$, the function $y = -(25 - x^2)^{1/2}$ works for $x \in (-5, 5)$.

For the function in question, that all works fine at most points on the circle. However, a problem occurs at both $(5, 0)$ and $(-5, 0)$. Neither point allows us to define a function $y(x)$ on an open interval containing $x = \pm 5$. The points $x = \pm 5$ cannot be in the interior of the domain of y.

This is connected with the fact that the graph becomes vertical at those two points.

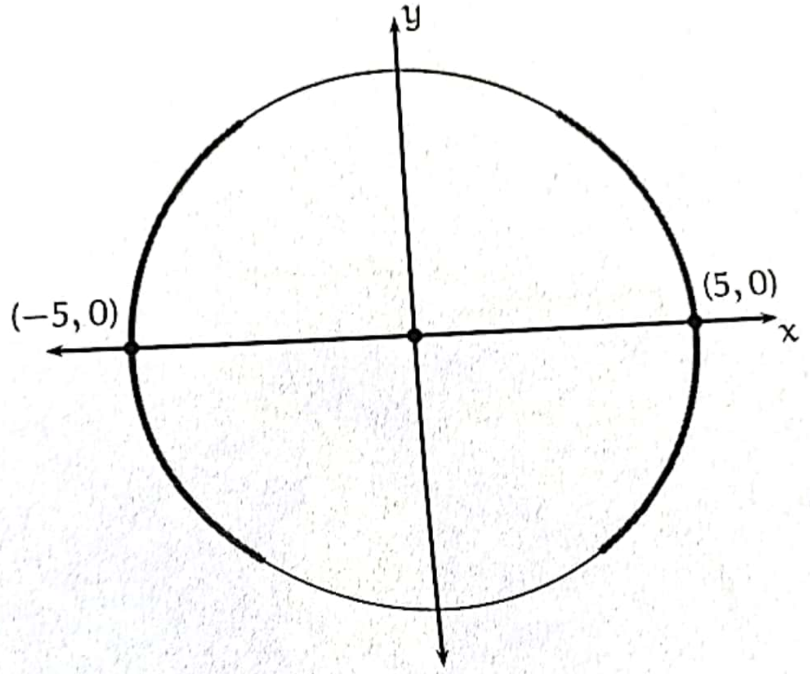


Figure 15.2.2: The circle is the graph of $x^2 + y^2 = 25$, which tries to implicitly define y as a function of x. The points $(5, 0)$ and $(-5, 0)$ pose particular problems as we are unable to write y as a function of x on a neighborhood of $x = \pm 5$ due to the verticality of the graph of y at $x = \pm 5$.

◂

15.3 The Implicit Function Theorem for $\mathbb{R}^2$

The key result is the Implicit Function Theorem. Here is a version for $\mathbb{R}^2$. The condition $(\partial G/\partial y)(x_0, y_0) \neq 0$ rules out vertical graphs at (x_0, y_0).

Implicit Function Theorem for $\mathbb{R}^2$. *Let $G(x, y)$ be a $\mathcal{C}^1$ function on a neighborhood of $(x_0, y_0) \in \mathbb{R}^2$. Suppose that $G(x_0, y_0) = c$. If*

$$\frac{\partial G}{\partial y}(x_0, y_0) \neq 0,$$

there exists a $\mathcal{C}^1$ function $y(x)$ defined on an interval I containing x_0 such that:

(a) $G(x, y(x)) = c$ *for all* $x \in I$,

(b) $y(x_0) = y_0$, *and*

(c) *The function y obeys*

$$y'(x_0) = -\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)}. \tag{15.3.2}$$

Point (c) follows from the Chain Rule once we establish that $y(x)$ is $\mathcal{C}^1$. To see this, just differentiate $G(x, y(x)) = 0$. We then obtain

$$\frac{\partial G}{\partial x}(x_0, y_0) + \left[\frac{\partial G}{\partial y}(x_0, y_0)\right] y'(x_0) = 0.$$

Rearrange to obtain equation (15.3.2).

15.4 Using the Implicit Function Theorem

► **Example 15.4.1: The Implicit Function Theorem and the Circle.** How does this apply to the circle $x^2 + y^2 = 25$ we studied in Example 15.2.1? Here we set $G(x, y) = x^2 + y^2$ and $c = 25$. Let's try $(x_0, y_0) = (3, -4)$ and see what happens.

Here

$$\frac{\partial G}{\partial y}(3, -4) = -8 \neq 0,$$

so we can apply the Implicit Function Theorem to find $y(x)$ solving $x^2 + [y(x)]^2 = 25$ with $y(3) = -4$ and $y'(3) = -2(3)/2(-4) = 3/4$. Compare to the solution y_1 given by $y_1(x) = -(25 - x^2)^{1/2}$. Then

$$y_1'(x) = -(1/2)(25 - x^2)^{-1/2}(2x) = \frac{x}{\sqrt{25 - x^2}}.$$

Then $y_1'(3) = 3/4$, exactly as with y. ◄

The Implicit Function Theorem will be useful for writing one set of economic variables as a function of other variable. For instance, suppose we solve the consumer's problem for prices p and income m. Can we write the demands for x and y as functions of prices and income? Once we characterize the solution via first order and second order equations, we will be able to use the Implicit Function Theorem to find whether we have proper demand functions.

15.5 Implicit Function Theorem: Sketch of Proof

Although we won't dot every i and cross every t, we will cover the basic idea behind the Implicit Function Theorem.

Sketch of Proof. By replacing G by $G(x, y) - c$, we may assume $G(x_0, y_0) = 0$. Also $(\partial G/\partial y)(x_0, y_0) \neq 0$ by hypothesis. We may also assume $(\partial G/\partial y)(x_0, y_0) > 0$ (otherwise, replace G by $-G$).

Now $G \in \mathcal{C}^1$, so $\frac{\partial G}{\partial y}$ is continuous. It follows that for $\varepsilon > 0$ small enough, $\partial G/\partial y > 0$ on the square S defined by

$$S = \{(x, y) : x_0 - \varepsilon \le x \le x_0 + \varepsilon, y_0 - \varepsilon \le y \le y_0 + \varepsilon\}.$$

Keeping in mind that $\partial G/\partial y$ is bounded above zero on the compact set S (Weierstrass) and that $G(x_0, y_0) = 0$, we may choose $\varepsilon_0 > 0$ small enough that $G(x, y_0 - \varepsilon) < 0 < G(x, y_0 + \varepsilon)$ whenever $|x - x_0| \le \varepsilon_0$. Now define the rectangle R by

$$R = \{(x, y) : |x - x_0| \le \varepsilon_0, |y - y_0| \le \varepsilon\}.$$

On R, $\partial G/\partial y > 0$. Moreover,$(x, y_0 \pm \varepsilon) \in R$ with $G(x, y_0 - \varepsilon) < 0$ and $G(x, y_0 + \varepsilon) > 0$.

Set $I = [x_0 - \varepsilon_0, x_0 + \varepsilon_0]$. Now G is continuous on R and takes opposite signs at the top and bottom of R. For each $x \in I$ the Intermediate Value Theorem yields a $y(x)$ with $G(x, y(x)) = 0$. Moreover, because G is increasing in y on R, there is only one such point $y(x)$ for each $x \in I$.

Now suppose $x_n \to x$ with $x_n \in I$. Because I is a closed interval, $x \in I$. Now consider $y_n = y(x_n)$. Because R is a compact interval, there is a subsequence (x_{n_k}, y_{n_k}) that converges to a point of R. Since $x_n \to x$, $x_{n_k} \to x$, and the only question is what is $y = \lim_k y_{n_k}$. We know $G(x_{n_k}, y_{n_k}) = 0$ and G is continuous, so $G(x, y) = 0$. Since this equation has a unique solution in R, $y = y(x)$, $y_{n_k} \to y(x)$. This means that all convergent subsequences of $y(x_n)$ have limit $y(x)$, so $\lim_n y(x_n) = y(x)$, showing that y is continuous.

The only thing left to do is show that y is a $\mathcal{C}^1$ function on I^0. As this is more technical than illuminating, we will skip that step. ∎

Section 6.3 – Basic Integration Rules

The notation $\int f(x)dx$ is used for an antiderivative of f and called an **indefinite integral.**

$$\int f(x)\,dx = F(x) \text{ means } F'(x) = f(x)$$

In general, to find $\int f(x)dx$, we find an antiderivative of $f(x)$, say $F(x)$, and then we write the indefinite integral as $\int f(x)dx = F(x) + C$. Here, C is called the **constant of integration.**

If given $\int_a^b f(x)dx$, this is a **definite integral** and to evaluate we'll use Part 2 of the

Fundamental Theorem of Calculus: $\int_a^b f(x)dx = F(b) - F(a)$

The Constant Rule for Integrals

$\int k dx = k \cdot x + C$, where k is a constant number.

Example 1: Find of each of the following integrals.

a. $\int 10 dx$

b. $\int_1^4 \pi dx$

The Power Rule for Integrals

$\int x^r dx = \frac{x^{r+1}}{r+1} + C$, where $r \neq -1$.

Example 2: Find of each of the following integrals.

a. $\int x^4 dx$

b. $\int_0^1 \frac{1}{\sqrt{x}} dx$

The Constant Multiple of a Function for Integrals

$\int k \cdot f(x)dx = k \int f(x)dx$, where k is a constant number.

Example 3: $\int_{-1}^{1} 4x^7 dx$

The Sum/Difference of Functions for Integrals

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

Example 4: $\int_0^1 (5x^4 + 4x + 7)dx$

Example 5: $\int \frac{x+2\sqrt{x}}{\sqrt{x}}dx$

Example 6:

a. $\int \sqrt[3]{x}(x-4)dx$

b. $\int 2x(x-1)(x+1)dx$

c. $\int \frac{5x^3+x}{x^3}dx$

Other times we are given the derivative and an initial value and we are asked to find the **original function.**

Example 7: Given $f'(x) = 2x + 2,\ f(1) = 5$, find $f(x)$.

Example 8: Given $f''(x) = 6x + 2,\ f'(0) = 2,\ f(0) = 10$, find $f(x)$.

E

a.

Integrals of Basic Trigonometric Functions:

$\int \sin x\, dx = -\cos x + C$

$\int \cos x\, dx = \sin x + C$

$\int \sec^2 x\, dx = \tan x + C$

$\int \csc^2 x\, dx = -\cot x + C$

$\int \sec x \tan x\, dx = \sec x + C$

$\int \csc x \cot x\, dx = -\csc x + C$

Example 9: $\int \sin x\,(\csc x + \cot x)\, dx$

Example 10: $\int_0^{\pi/3} \sec x \tan x\, dx$

Example 11: Given $f'(x) = -3\sin x$, $f(\pi) = -1$, find $f(x)$.

Integral of $\frac{1}{x}$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Example 12: $\int_1^2 \frac{x^2-2}{x}\,dx$

Integrals of Exponential Functions

$$\int e^x dx = e^x + C$$

$\int a^x dx = \frac{a^x}{\ln a} + C$, where $a > 0,\ a \neq 1$

Example 13: $\int (2^x + 5e^x)dx$

Integrals of the Hyperbolic Functions

$\int \sinh x \, dx = \cosh x + C$

$\int \cosh x \, dx = \sinh x + C$

Recall: $\cosh x = \frac{e^x + e^{-x}}{2}$; $\sinh x = \frac{e^x - e^{-x}}{2}$

Example 14: $\int_0^1 2 \sinh x \, dx$

Integrals Resulting in Inverse Trigonometric Functions

$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

$\int \frac{1}{1+x^2} dx = \arctan x + C$

$\int \frac{1}{|x|\sqrt{x^2-1}} dx = \operatorname{arcsec} x + C$

Example 15: $\int_{1/2}^{\sqrt{2}/2} \frac{2}{\sqrt{1-x^2}} dx$

Integrating Piece-wise Defined Functions

Example 16: Let $f(x) = \begin{cases} x+2, & -2 \le x \le 0 \\ 2, & 0 < x \le 1 \\ 4-2x, & 1 < x \le 2 \end{cases}$

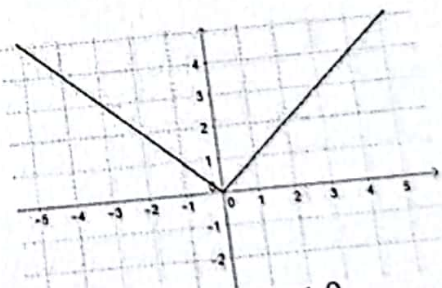
Note how the function changes over the specified domain!

Set-up the integral needed to integrate $\int_{-2}^{2} f(x)dx$.

Integrals Involving Absolute Value

Example 17: $\int_{-1}^{2} |x| dx$

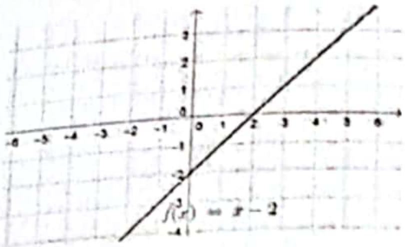
Recall that $y = |x|$ is a piecewise function!



$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \ge 0 \end{cases}$$

Example 18:

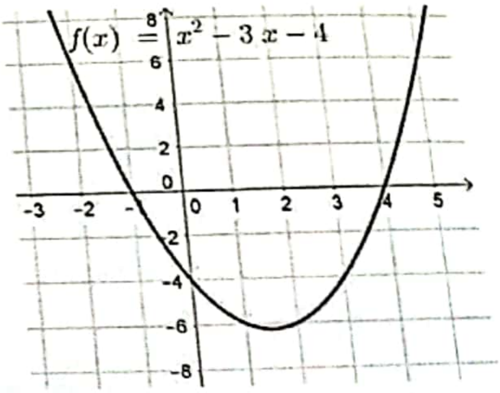
a. $\int_1^4 |x-2|dx$



b. $\int_5^6 |x-2|dx$

Example 19: Set up the following integrals.

a. $\int_1^5 |x^2 - 3x - 4|\, dx$



b. $\int_1^5 |x^2 + 4|\, dx$

CHAPTER 7

Techniques of Integration

§7.1. Substitution

Integration, unlike differentiation, is more of an art-form than a collection of algorithms. Many problems in applied mathematics involve the integration of functions given by complicated formulae, and practitioners consult a *Table of Integrals* in order to complete the integration. There are certain methods of integration which are essential to be able to use the Tables effectively. These are: substitution, integration by parts and partial fractions. In this chapter we will survey these methods as well as some of the ideas which lead to the tables. After the examination on this material, students will be free to use the Tables to integrate.

The idea of substitution was introduced in section 4.1 (recall Proposition 4.4). To integrate a differential $f(x)dx$ which is not in the table, we first seek a function $u = u(x)$ so that the given differential can be rewritten as a differential $g(u)du$ which does appear in the table. Then, if $\int g(u)du = G(u) + C$, we know that $\int f(x)dx = G(u(x)) + C$. Finding and employing the function u often requires some experience and ingenuity as the following examples show.

Example 7.1 $\int x\sqrt{2x+1}dx = ?$

Let $u = 2x+1$, so that $du = 2dx$ and $x = (u-1)/2$. Then

$$\int x\sqrt{2x+1}dx = \int \tfrac{u-1}{2}u^{1/2}\tfrac{du}{2} = \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right) du = \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \tag{7.1}$$

$$= \tfrac{1}{30}u^{3/2}(3u-5) + C = \frac{1}{30}(2x+1)^{3/2}(6x-2) + C = \frac{1}{15}(2x+1)^{3/2}(3x-1) + C\,, \tag{7.2}$$

where at the end we have replaced u by $2x+1$.

Example 7.2 $\int \tan x dx = ?$

Since this isn't on our tables, we revert to the definition of the tangent: $\tan x = \sin x/\cos x$. Then, letting $u = \cos x$, $du = -\sin x dx$ we obtain

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} = -\ln u + C = -\ln \cos x + C = \ln \sec x + C. \tag{7.3}$$

Example 7.3 $\int \sec x dx = ?$.

This is tricky, and there are several ways to find the integral. However, if we are guided by the principle of rewriting in terms of sines and cosines, we are led to the following:

$$\sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x}. \tag{7.4}$$

Now we can try the substitution $u = \sin x$, $du = \cos x dx$. Then

$$\int \sec x dx = \int \frac{du}{1-u^2}. \tag{7.5}$$

This looks like a dead end, but a little algebra pulls us through. The identity

$$\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right) \tag{7.6}$$

leads to

$$\int \frac{du}{1-u^2} dx = \frac{1}{2}\int\left(\frac{1}{1+u} + \frac{1}{1-u}\right) du = \frac{1}{2}(\ln(1+u) - \ln(1-u)) + C. \tag{7.7}$$

Using $u = \sin x$, we finally end up with

$$\int \sec x dx = \frac{1}{2}(\ln(1+\sin x) - \ln(1-\sin x) + C) = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right) + C. \tag{7.8}$$

Example 7.4 As a circle rolls along a horizontal line, a point on the circle traverses a curve called the *cycloid*. A *loop* of the cycloid is the trajectory of a point as the circle goes through one full rotation. Let us find the length of one loop of the cycloid traversed by a circle of radius 1.

Let the variable t represent the angle of rotation of the circle, in radians, and start (at $t = 0$) with the point of intersection P of the circle and the line on which it is rolling. After the circle has rotated through t radians, the position of the point is as given in figure 7.1. The point of contact of the circle with the line is now t units to the right of the original point of contact (assuming no slippage), so

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t. \tag{7.9}$$

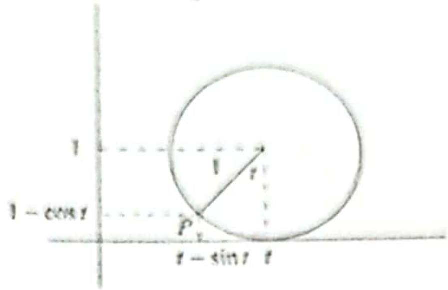
To find arc length, we use $ds^2 = dx^2 + dy^2$, where $dx = (1-\cos t)dt$, $dy = \sin t dt$. Thus

$$ds^2 = ((1-\cos t)^2 + \sin^2 t)^2 dt^2 = (2 - 2\cos t)^2 dt^2 \tag{7.10}$$

so $ds = \sqrt{2(1-\cos t)}dt$, and the arc length is given by the integral

$$L = \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t}dt. \tag{7.11}$$

Figure 7.1



To evaluate this integral by substitution, we need a factor of $\sin t$. We can get this by multiplying and dividing by $\sqrt{1+\cos t}$:

(7.12) $$\sqrt{1-\cos t} = \frac{\sqrt{1-\cos^2 t}}{\sqrt{1+\cos t}} = \frac{|\sin t|}{\sqrt{1+\cos t}} .$$

By symmetry around the line $t = \pi$, the integral will be twice the integral from 0 to π. In that interval, $\sin t$ is positive, so we can drop the absolute value signs. Now, the substitution $u = \cos t$, $du = -\sin t dt$ will work. When $t = 0$, $u = 1$, and when $t = \pi, u = -1$. Thus

(7.13) $$L = -2\sqrt{2}\int_1^{-1} u^{-1/2}du = 2\sqrt{2}\int_{-1}^{1} u^{-1/2}du = 2\sqrt{2}(2u^{1/2})\Big|_{-1}^{1} = 8\sqrt{2} .$$

§7.2. Integration by Parts

Sometimes we can recognize the differential to be integrated as a product of a function which is easily differentiated and a differential which is easily integrated. For example, if the problem is to find

(7.14) $$\int x\cos x dx$$

then we can easily differentiate $f(x) = x$, and integrate $\cos x dx$ separately. When this happens, the integral version of the product rule, called *integration by parts*, may be useful, because it interchanges the roles of the two factors.

Recall the product rule: $d(uv) = udv + vdu$, and rewrite it as

(7.15) $$udv = d(uv) - vdu$$

In the case of 7.14, taking $u = x$, $dv = \cos x dx$, we have $du = dx$, $v = \sin x$. Putting this all in 7.15:

(7.16) $$x\cos x dx = d(x\sin x) - \sin x dx ,$$

and we can easily integrate the right hand side to obtain

(7.17)
$$\int x\cos x dx = x\sin x - \int \sin x dx = x\sin x + \cos x + C.$$

Proposition 7.1 (Integration by Parts) *For any two differentiable functions u and v:*

(7.18)
$$\int u dv = uv - \int v du.$$

To integrate by parts:
1. First identify the parts by reading the differential to be integrated as the product of a function u easily differentiated, and a differential dv easily integrated.
2. Write down the expressions for u, dv and du, v.
3. Substitute these expressions in 7.18.
4. Integrate the new differential vdu.

Example 7.5 Find $\int xe^x dx$.

Let $u = x$, $dv = e^x dx$. Then $du = dx$, $v = e^x$. 7.18 gives us

(7.19)
$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

Example 7.6 Find $\int x^2 e^x dx$.

The substitution $u = x^2$, $dv = e^x dx$, $du = 2x dx$, $v = e^x$ doesn't immediately solve the problem, but reduces us to example 3:

(7.20)
$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx = x^2 e^x - 2(xe^x - e^x + C) = x^2 e^x - 2xe^x + 2e^x + C.$$

Example 7.7 To find $\int \ln x dx$, we let $u = \ln x$, $dv = dx$, so that $du = (1/x)dx$, $v = x$, and

(7.21)
$$\int \ln x dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + C.$$

This same idea works for $\arctan x$: Let

(7.22)
$$u = \arctan x,\ dv = dx \quad du = \frac{dx}{1+x^2},\ v = x,$$

and thus

(7.23)
$$\int \arctan x = x\arctan x - \int \frac{x}{1+x^2}dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C,$$

where the last integration is accomplished by the new substitution $u = 1 + x^2$, $du = 2xdx$.

Example 7.8 These ideas lead to some clever strategies. Suppose we have to integrate $e^x \cos x dx$. We see that an integration by parts leads us to integrate $e^x \sin x dx$, which is just as hard. But suppose we integrate by parts again? See what happens:

Letting $u = e^x$, $dv = \cos x dx$, $du = e^x dx, v = \sin x$, we get

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx . \tag{7.24}$$

Now integrate by parts again: letting $u = e^x$, $dv = \sin x dx$, $du = e^x dx, v = -\cos x$, we get

$$\int e^x \sin x dx = e^x \cos x + \int e^x \cos x dx . \tag{7.25}$$

Inserting this in 7.24 leads to

$$\int e^x \cos x dx = e^x \sin x - e^x \cos x - \int e^x \cos x dx . \tag{7.26}$$

Bringing the last term over to the left hand side and dividing by 2 gives us the answer:

$$\int e^x \cos x dx = \frac{1}{2}(e^x \sin x - e^x \cos x) + C . \tag{7.27}$$

Example 7.9 If a calculation of a definite integral involves integration by parts, it is a good idea to evaluate as soon as integrated terms appear. We illustrate with the calculation of

$$\int_1^4 \ln x dx \tag{7.28}$$

Let $u = \ln x dx$, $dv = dx$ so that $du = dx/x, v = x$, and

$$\int_1^4 \ln x dx = x \ln x\Big|_1^4 - \int_1^4 dx = 4\ln 4 - x\Big|_1^4 = 4\ln 4 - 3 . \tag{7.29}$$

Example 7.10

$$\int_0^{1/2} \arcsin x dx = \ ? . \tag{7.30}$$

We make the substitution $u = \arcsin x$, $dv = dx$, $du = dx/\sqrt{1-x^2}$, $v = x$. Then

$$\int_0^{1/2} \arcsin x dx = x \arcsin x\Big|_0^{1/2} - \int_0^{1/2} \frac{x dx}{\sqrt{1-x^2}} . \tag{7.31}$$

Now, to complete the last integral, let $u = 1 - x^2$, $du = -2x dx$, leading us to

$$\int_0^{1/2} \arcsin x dx = \frac{1}{2}\left(\frac{\pi}{6}\right) + \frac{1}{2}\int_1^{3/4} u^{-1/2} du = \frac{\pi}{12} + u^{1/2}\Big|_1^{3/4} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 . \tag{7.32}$$

§7.3. Partial Fractions

The point of the partial fractions expansion is that integration of a rational function can be reduced to the following formulae, once we have determined the roots of the polynomial in the denominator.

Proposition 7.2

a) $\int \frac{dx}{x-a} = \ln|x-a| + C,$

b) $\int \frac{du}{u^2+b^2} = \frac{1}{b}\arctan\left(\frac{u}{b}\right) + C,$

c) $\int \frac{u\,du}{u^2+b^2} = \frac{1}{2}\ln(u^2+b^2) + C.$

These are easily verified by differentiating the right hand sides (or by using previous techniques).

Example 7.11 Let us illustrate with an example we've already seen. To find the integral

(7.33)
$$\int \frac{dx}{(x-a)(x-b)}$$

we check that

(7.34)
$$\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right),$$

so that

(7.35)
$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b}(\ln|x-a| - \ln|x-b|) + C = \frac{1}{a-b}\ln\left|\frac{x-a}{x-b}\right| + C.$$

The trick 7.34 can be applied to any rational function. Any polynomial can be written as a product of factors of the form $x-r$ or $(x-a)^2+b^2$, where r is a real root and the quadratic terms correspond to the conjugate pairs of complex roots. The partial fraction expansion allows us to write the quotient of polynomials as a sum of terms whose denominators are of these forms, and thus the integration is reduced to Proposition 7.2.

Here is the partial fractions procedure.

1. Given a rational function $R(x)$, if the degree of the numerator is not less than the degree of the denominator, by long division, we can write

(7.36)
$$R(x) = Q(x) + \frac{p(x)}{q(x)}$$

where now $\deg p < \deg q$.

2. Find the roots of $q(x) = 0$. If the roots are all distinct (there are no multiple roots), write p/q as a sum of terms of the form

(7.37)
$$\frac{A}{x-r}, \quad \frac{B}{(x-a)^2+b^2}, \quad \frac{Cx}{(x-a)^2+b^2}.$$

3. Find the values of A, B, C, ...
4. Integrate term by term using Proposition 7.2.

If the roots are not distinct, the expansion is more complicated; we shall resume this discussion later. For the present let us concentrate on the case of distinct roots, and how to find the coefficients A, B, C in 7.37.

Example 7.12 Integrate $\int \frac{xdx}{(x-1)(x-2)}$.

First we write

$$\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}. \tag{7.38}$$

Now multiply this equation by $(x-1)(x-2)$, getting

$$x = A(x-2) + B(x-1). \tag{7.39}$$

If we substitute $x = 1$, we get $1 = A(1-2)$, so $A = -1$; now letting $x = 2$, we get $2 = B(2-1$, so $B = 2$, and 7.38 becomes

$$\frac{x}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{2}{x-2}. \tag{7.40}$$

Integrating, we get

$$\int \frac{xdx}{(x-1)(x-2)} = -\ln|x-1| + 2\ln|x-2| + C = \ln\frac{(x-2)^2}{|x-1|} + C. \tag{7.41}$$

So, this is the procedure for finding the coefficients of the partial fractions expansion when the roots are all real and distinct:

1. Write down the expansion with unknown coefficients.
2. Multiply through by the product of all the terms $x - r$.
3. Substitute each root in the above equation; each substitution determines one of the coefficients.

Example 7.13 Integrate $\int \frac{(x^2-3)dx}{(x^2-1)(x-3)}$.

Here the roots are $\pm 1, 3$, so we have the expansion

$$\frac{x^2-3}{(x^2-1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-3} \tag{7.42}$$

leading to

$$x^2 - 3 = A(x-1)(x-3) + B(x+1)(x-3) + C(x+1)(x-1). \tag{7.43}$$

Substitute $x = -1$: $1 - 3 = A(-2)(-4)$, so $A = -1/4$.
Substitute $x = 1$: $1 - 3 = B(2)(-2)$, so $B = 1/2$.
Substitute $x = 3$: $9 - 3 = C(4)(2)$, so $C = 3/4$, and 7.42 becomes

$$\frac{x^2-3}{(x^2-1)(x-3)} = \left(-\frac{1}{4}\right)\frac{1}{x+1} + \left(\frac{1}{2}\right)\frac{1}{x-1} + \left(\frac{3}{4}\right)\frac{1}{x-3}, \tag{7.44}$$

and the integral is

$$\int \frac{(x^2-3)dx}{(x^2-1)(x-3)} = -\frac{1}{4}\ln|x+1| + \frac{1}{2}\ln|x-1| + \frac{3}{4}\ln|x-3| + C. \tag{7.45}$$

§7.3.1 Quadratic Factors

Example 7.14 $\int \frac{dx}{x^2-4x-5} = ?$

Here we can factor: $x^2-4x-5=(x+1)(x-5)$, so we can write

(7.46)
$$\frac{1}{x^2-4x-5}=\frac{A}{x+1}+\frac{B}{x-5}$$

and solve for A and B as above: $A=1/6$, $B=-1/6$, so we have

(7.47)
$$\frac{1}{x^2-4x-5}=\frac{1}{6}\left(\frac{1}{x-5}-\frac{1}{x+1}\right)$$

and the integral is

(7.48)
$$\int\frac{dx}{x^2-4x-5}=\frac{1}{6}\ln\left|\frac{x-5}{x+1}\right|+C.$$

Example 7.15 $\int \frac{dx}{x^2-4x+5} = ?$

Here we can't find real factors, because the roots are complex. But we can complete the square: $x^2-4x+5=(x-2)^2+1$, and now use Proposition 7.2b:

(7.49)
$$\int\frac{dx}{x^2-4x+5}=\int\frac{dx}{(x-2)^2+1}=\arctan(x-2)+C.$$

Example 7.16 $\int \frac{(x+3)dx}{x^2-4x+5} = ?.$

Here we have to be a little more resourceful. Again, we complete the square, giving

(7.50)
$$\frac{x+3}{x^2-4x+5}=\frac{x+3}{(x-2)^2+1}.$$

If only that $x+3$ were $x-2$, we could use Proposition 7.2c, with $u=x-2$. Well, since $x+3=x-2+5$, there is no problem:

(7.51)
$$\int\frac{(x+3)dx}{x^2-4x+5}=\int\frac{(x-2)dx}{(x-2)^2+1}+\int\frac{5dx}{(x-2)^2+1}=$$
$$\frac{1}{2}\ln((x-2)^2+1)+5\arctan(x-2)+C.$$

Example 7.17 $\int \frac{(2x+1)dx}{x^2-6x+14} = ?.$

First, we complete the square in the denominator: $x^2-6x+14=(x-3)^2+5$. Now, write the numerator in terms of $x-3$: $2x+1=2(x-3)+7$. This gives the expansion:

(7.52)
$$\frac{(2x+1)dx}{x^2-6x+14}=\frac{7}{x^2-6x+14}+2\frac{x-3}{x^2-6x+14}$$

so, using Proposition 7.2:

(7.53)
$$\int \frac{(2x+1)dx}{x^2-6x+14} = 7\int \frac{dx}{(x-3)^2+5} + 2\int \frac{(x-3)dx}{(x-3)^2+5}$$

(7.54)
$$= \frac{7}{\sqrt{5}}\arctan\frac{x-3}{\sqrt{5}} + \ln((x-3)^2+5) + C\,.$$

Example 7.18 $\int \frac{(x+1)dx}{x(x^2+1)} = ?.$

Here we have to expect each of the terms in Proposition 7.2 to appear, so we try an expression of the form

(7.55)
$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{B}{x^2+1} + \frac{Cx}{x^2+1}\,.$$

Clearing the denominators on the right, we are led to the equation

(7.56)
$$x+1 = A(x^2+1) + Bx + Cx^2\,.$$

Setting $x=0$ gives $1=A$. But we have no more roots to substitute to find B and C, so instead we equate coefficients. The coefficient of x^2 on the left is 0, and on the right is $A+C$, so $A+C=0$; since $A=1$, we learn that $C=-1$. Comparing coefficients of x we learn that $1=B$. Thus 7.55 becomes

(7.57)
$$\frac{x+1}{x(x^2+1)} = \frac{1}{x} + \frac{1}{x^2+1} - \frac{x}{x^2+1}\,,$$

and our integral is

(7.58)
$$\int \frac{(x+1)dx}{x(x^2+1)} = \ln|x| + \arctan x - \frac{1}{2}\ln(x^2+1) + C\,.$$

Example 7.19 $\int \frac{(x^2+1)dx}{x(x^2-4x+5)} = ?.$

The denominator is $x((x-2)^2+1)$, so we expect a partial fractions expansion of the form

(7.59)
$$\frac{x^2+1}{x(x^2-4x+5)} = \frac{A}{x} + \frac{B}{(x-2)^2+1} + \frac{C(x-2)}{(x-2)^2+1}\,.$$

Clearing of denominators, we obtain the equation

(7.60)
$$x^2+1 = A((x-2)^2+1) + Bx + C(x-2)x\,.$$

For $x=0$, we obtain $1=A(5)$, so $A=1/5$. Comparing coefficients of x^2 we obtain $1=A+C$, so $C=-1/5$. Comparing coefficients of x we obtain $0=-4A+B-2C$, so $0=-4/5+B+2/5$, so $B=2/5$ and 7.59 becomes

(7.61)
$$\frac{x^2+1}{x(x^2-4x+5)} = \left(\frac{1}{5}\right)\frac{1}{x} + \left(\frac{2}{5}\right)\frac{1}{(x-2)^2+1} - \left(\frac{1}{5}\right)\frac{x-2}{(x-2)^2+1}\,,$$

which we can integrate to

(7.62)
$$\int \frac{(x^2+1)dx}{x(x^2-4x+5)} = \frac{1}{5}\ln|x| + \frac{2}{5}\arctan(x-2) - \frac{1}{10}\ln(x^2-4x+5) + C.$$

Multiple Roots

If the denominator has a multiple root, that is there is a factor $x-r$ raised to a power, then we have to allow for the possibility of terms in the partial fraction of the form $1/(x-r)$ raised to the same power. But then the numerator can be (as we have seen above in the case of quadratic factors) a polynomial of degree as much as one less than the power. This is best explained through a few examples.

Example 7.20 $\int \frac{(x^2+1)dx}{x^3(x-1)} = ?.$

We have to allow for the possibility of a term of the form $(Ax^2+Bx+C)/x^3$, or, what is the same, an expansion of the form

(7.63)
$$\frac{x^2+1}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1}.$$

Clearing of denominators, we obtain

(7.64)
$$x^2+1 = Ax^2(x-1) + Bx(x-1) + C(x-1) + Dx^3.$$

Substituting $x=0$ we obtain $1=C(-1)$, so $C=-1$. Substituting $x=1$, we obtain $2=D$. To find A and B we have to compare coefficients of powers of x. Equating coefficients of x^3, we have $0=A+D$, so $A=-2$. Equating coefficients of x^2, we have $1=-A+B$, so $B=1+A=-1$. Thus the expansion 7.63 is

(7.65)
$$\frac{x^2+1}{x^3(x-1)} = -\frac{2}{x} - \frac{1}{x^2} - \frac{1}{x^3} + \frac{2}{x-1},$$

which we can integrate term by term:

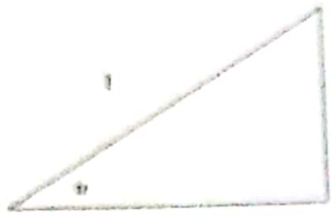
(7.66)
$$\int \frac{(x^2+1)dx}{x^3(x-1)} = -2\ln|x| + \frac{1}{x} + \frac{1}{2x^2} + 2\ln|x-1| + C.$$

§7.4. Trigonometric Methods

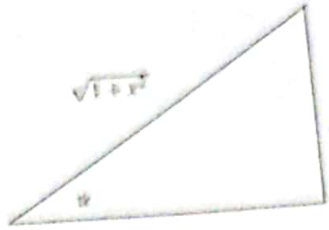
Now, although the above techniques are all that one needs to know in order to use a Table of Integrals, there is one form which appears so often, that it is worthwhile seeing how the integration formulae are found. Expressions involving the square root of a quadratic function occur quite frequently in practice. How do we integrate $\sqrt{1-x^2}$ or $\sqrt{1+x^2}$?

When the expressions involve a square root of a quadratic, we can convert to trigonometric functions using the substitutions suggested by figure 7.2.

Figure 7.2



(A)



(B)

Example 7.21 To find $\int\sqrt{1-x^2}dx$, we use the substitution of figure 7.2A: $x=\sin u$, $dx=\cos u\,du$, $\sqrt{1-x^2}=\cos u$. Then

(7.67)
$$\int\sqrt{1-x^2}dx=\int\cos^2 u\,du\,.$$

Now, we use the half-angle formula: $\cos^2 u=(1+\cos 2u)/2$:

(7.68)
$$\int\sqrt{1-x^2}dx=\int\frac{1+\cos 2u}{2}du=\frac{u}{2}+\frac{\sin 2u}{4}+C\,.$$

Now, to return to the original variable x, we have to use the double angle formula: $\sin 2u=2\sin u\cos u=x\sqrt{1-x^2}$, and we finally have the answer:

(7.69)
$$\int\sqrt{1-x^2}dx=\frac{\arcsin x}{2}+\frac{x\sqrt{1-x^2}}{4}+C\,.$$

Example 7.22 To find $\int\sqrt{1+x^2}dx$, we use the substitution of figure 7.2B: $x=\tan u$, $dx=\sec^2 u\,du$, $\sqrt{1+x^2}=\sec u$. Then

(7.70)
$$\int\sqrt{1+x^2}dx=\int\sec^3 u\,du\,.$$

This is still a hard integral, but we can discover it by an integration by parts (see Practice Problem set 4, problem 6) to be

(7.71)
$$\int\sec^3 du=\frac{1}{2}(\sec u\tan u+\ln|\sec u+\tan u|)+C\,.$$

Now, we return to figure 7.2B to write this in terms of x: $\tan u=x$, $\sec u=\sqrt{1+x^2}$. We finally obtain

(7.72)
$$\int\sqrt{1+x^2}dx=\frac{1}{2}(x\sqrt{1+x^2}+\ln|\sqrt{1+x^2}+x|)+C\,.$$

Example 7.23 $\int x\sqrt{1+x^2}dx=\ ?$.

Don't be misled: always try simple substitution first; in this case the substitution $u = 1 + x^2$, $du = 2xdx$ leads to the formula

(7.73)
$$\int x\sqrt{1+x^2}dx = \frac{1}{2}\int u^{1/2}du = \frac{2}{3}(1+x^2)^{3/2} + C.$$

Example 7.24 $\int x^2\sqrt{1-x^2}dx = ?$.

Here simple substitution fails, and we use the substitution of figure 7.2A: $x = \sin u$, $dx = \cos u du$, $\sqrt{1-x^2} = \cos u$. Then

(7.74)
$$\int x^2\sqrt{1-x^2}dx = \int \sin^2 u \cos^2 u du.$$

This integration now follows from use of double- and half-angle formulae:

(7.75)
$$\int \sin^2 u \cos^2 u du = \frac{1}{4}\int \sin^2(2u)\,du = \frac{1}{8}\int (1-\cos(4u))\,du = \frac{1}{8}\left(u - \frac{\sin(4u)}{4}\right) + C.$$

Now, $\sin(4u) = 2\sin(2u)\cos(2u) = 4\sin u \cos u(1-2\sin^2 u) = 4x\sqrt{1-x^2}(1-2x^2)$. Finally

(7.76)
$$\int x^2\sqrt{1-x^2}dx = \frac{\arcsin x}{8} + \frac{x\sqrt{1-x^2}(1-2x^2)}{2} + C.$$

For the remainder of this course, we shall assume that you have a table of integrals available, and know how to use it. There are several handbooks, and every Calculus text has a table of integrals on the inside back cover. There are a few tables on the web:

```
http://math2.org/math/integrals/tableof.htm
http://www.cahs1.org/lessonIcalc/table
   /table_of_integrals.htm
http://www.engineering.com/community/library/textbook
   /calculus/calculus_table_integrals_content.htm
http://www.maths.abdn.ac.uk/~jrp/ma1002/website/int
   /node51.html
```

Chapter 4
Partial Fractions

4.1 Introduction: A fraction is a symbol indicating the division of integers. For example, $\frac{13}{9}, \frac{2}{3}$ are fractions and are called Common Fraction. The dividend (upper number) is called the numerator N(x) and the divisor (lower number) is called the denominator, D(x).

From the previous study of elementary algebra we have learnt how the sum of different fractions can be found by taking L.C.M. and then add all the fractions. For example

i) $\frac{1}{x-1}+\frac{2}{x+2} = \frac{3x}{(x-1)(x+2)}$

ii) $\frac{2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x-2} = \frac{9x^2+5x-3}{(x+1)^2(x-2)}$

Here we study the reverse process, i.e., we split up a single fraction into a number of fractions whose denominators are the factors of denominator of that fraction. These fractions are called **Partial fractions.**

4.2 Partial fractions :

To express a single rational fraction into the sum of two or more single rational fractions is called **Partial fraction resolution.**

For example,

$$\frac{2x+x^2-1}{x(x^2-1)} = \frac{1}{x}+\frac{1}{x-1}-\frac{1}{x+1}$$

$\frac{2x+x^2-1}{x(x^2-1)}$ is the resultant fraction and $\frac{1}{x}+\frac{1}{x-1}-\frac{1}{x+1}$ are its partial fractions.

4.3 Polynomial:

Any expression of the form $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots.. + a_2x^2 + a_1x + a_0$ where $a_n, a_{n-1}, \ldots.., a_2, a_1, a_0$ are real constants, if $a_n \neq 0$ then P(x) is called polynomial of degree n.

4.4 Rational fraction:

We know that $\frac{p}{q}$, $q \neq 0$ is called a rational number. Similarly the quotient of two polynomials $\frac{N(x)}{D(x)}$ where $D(x) \neq 0$, with no common factors, is called a rational fraction. A rational fraction is of two types:

4.5 Proper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$ is called a proper fraction if the degree of numerator N(x) is less than the degree of Denominator D(x).

For example

(i) $\frac{9x^2 - 9x + 6}{(x-1)(2x-1)(x+2)}$

(ii) $\frac{6x + 27}{3x^3 - 9x}$

4.6 Improper Fraction:

A rational fraction $\frac{N(x)}{D(x)}$ is called an improper fraction if the degree of the Numerator N(x) is greater than or equal to the degree of the Denominator D(x)

For example

(i) $\frac{2x^3 - 5x^2 - 3x - 10}{x^2 - 1}$

(ii) $\frac{6x^3 - 5x^2 - 7}{3x^2 - 2x - 1}$

Note: An improper fraction can be expressed, by division, as the sum of a polynomial and a proper fraction.

For example:

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x + 3) + \frac{8x - 4}{x^2 - 2x - 1}$$

Which is obtained as, divide $6x^2 + 5x^2 - 7$ by $3x^2 - 2x - 1$ then we get a polynomial (2x+3) and a proper fraction $\frac{8x - 4}{x^2 - 2x - 1}$

4.7 Process of Finding Partial Fraction:

A proper fraction $\frac{N(x)}{D(x)}$ can be resolved into partial fractions as:

(I) If in the denominator D(x) a linear factor (ax + b) occurs and is non-repeating, its partial fraction will be of the form $\frac{A}{ax + b}$, where A is a constant whose value is to be determined.

(II) If in the denominator D(x) a linear factor (ax + b) occurs n times, i.e., $(ax+b)^n$, then there will be n partial fractions of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\dots\dots+\frac{A_n}{(ax+b)^n}$$

,where A_1, A_2, A_3 ---------- A_n are constants whose values are to be determined

(III) If in the denominator D(x) a quadratic factor ax^2+bx+c occurs and is non-repeating, its partial fraction will be of the form $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants whose values are to be determined.

(IV) If in the denominator a quadratic factor ax^2+bx+c occurs n times, i.e., $(ax^2+bx+c)^n$,then there will be n partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+$$

$$\text{-----------}+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Where A_1, A_2, A_3 --------- A_n and B_1, B_2, B_3 ------- B_n are constants whose values are to be determined.

Note: The evaluation of the coefficients of the partial fractions is based on the following theorem:

If two polynomials are equal for all values of the variables, then the coefficients having same degree on both sides are equal, for example , if

$$px^2+qx+a=2x^2-3x+5 \quad \forall\, x \text{ , then}$$

$$p=2, \quad q=-3 \quad \text{and} \quad a=5.$$

4.8 Type I

When the factors of the denominator are all linear and distinct i.e., non repeating.

Example 1:

Resolve $\frac{7x-25}{(x-3)(x-4)}$ into partial fractions.

Solution:

$$\frac{7x-25}{(x-3)(x-4)}=\frac{A}{x-3}+\frac{B}{x-4} \text{-----------------}(1)$$

Multiplying both sides by L.C.M. i.e., $(x-3)(x-4)$, we get

$$7x-25=A(x-4)+B(x-3) \text{-------------} \quad (2)$$

$$7x-25=Ax-4A+Bx-3B$$

$$7x - 25 = Ax + Bx - 4A - 3B$$
$$7x - 25 = (A + B)x - 4A - 3B$$

Comparing the co-efficients of like powers of x on both sides, we have

$$7 = A + B \text{ and}$$
$$-25 = -4A - 3B$$

Solving these equation we get

$$A = 4 \quad \text{and} \quad B = 3$$

Hence the required partial fractions are:

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

Alternative Method:

Since $7x - 25 = A(x - 4) + B(x - 3)$

Put $x - 4 = 0, \Rightarrow x = 4$ in equation (2)

$$7(4) - 25 = A(4 - 4) + B(4 - 3)$$
$$28 - 25 = 0 + B(1)$$
$$B = 3$$

Put $x - 3 = 0 \Rightarrow x = 3$ in equation (2)

$$7(3) - 25 = A(3 - 4) + B(3 - 3)$$
$$21 - 25 = A(-1) + 0$$
$$-4 = -A$$
$$A = 4$$

Hence the required partial fractions are

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

Note : The R.H.S of equation (1) is the identity equation of L.H.S

Example 2:

write the identity equation of $\frac{7x-25}{(x-3)(x-4)}$

Solution : The identity equation of $\frac{7x-25}{(x-3)(x-4)}$ is

$$\frac{7x-25}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

Example 3:

Resolve into partial fraction: $\frac{1}{x^2 - 1}$

Solutios: $$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$1 = A(x+1) + B(x-1) \qquad (1)$$

Put $x - 1 = 0, \Rightarrow x = 1$ in equation (1)

$$1 = A(1+1) + B(1-1) \Rightarrow A = \frac{1}{2}$$

Put $x + 1 = 0, \Rightarrow x = -1$ in equation (1)

$$1 = A(-1+1) + B(-1-1)$$

$$1 = -2B, \Rightarrow B = \frac{1}{2}$$

$$\frac{1}{x^2-1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

Example 4:

Resolve into partial fractions $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}$

Solution:

This is an improper fraction first we convert it into a polynomial and a proper fraction by division.

$$\frac{6x^3+5x^2-7}{3x^2-2x-1} = (2x+3) + \frac{8x-4}{x^2-2x-1}$$

Let $$\frac{8x-4}{x^2-2x-1} = \frac{8x-4}{(3x+1)} = \frac{A}{x-1} + \frac{B}{3x+1}$$

Multiplying both sides by $(x-1)(3x+1)$ we get

$$8x - 4 = A(3x+1) + B(x-1) \qquad (I)$$

Put $x - 1 = 0, \Rightarrow x = 1$ in (I), we get

The value of A

$$8(1) - 4 = A(3(1)+1) + B(1-1)$$

$$8 - 4 = A(3+1) + 0$$

$$4 = 4A$$

$$\Rightarrow \quad A = 1$$

Put $3x + 1 = 0 \Rightarrow x = -\dfrac{1}{3}$ in (I)

$$8\left(-\frac{1}{3}\right)-4=B\left(-\frac{1}{3}-1\right)$$
$$-\frac{8}{3}-4=\left(-\frac{4}{3}\right)$$
$$-\frac{20}{3}=-\frac{4}{3}B$$
$$\Rightarrow \quad B=\frac{20}{3}x\ \frac{3}{4}\quad =5$$

Hence the required partial fractions are

$$\frac{6x^3+5x^2-7}{3x^2-2x-1}=(2x+3)+\frac{1}{x-1}+\frac{5}{3x+1}$$

Example 5:

Resolve into partial fraction $\dfrac{8x-8}{x^3-2x^2-8x}$

Solution: $$\frac{8x-8}{x^3-2x^2-8x}=\frac{8x-8}{x(x^2-2x-8)}=\frac{8x-8}{x(x-4)(x+2)}$$

Let $$\frac{8x-8}{x^3-2x^2-8x}=\frac{A}{x}+\frac{B}{x-4}+\frac{C}{x+2}$$

Multiplying both sides by L.C.M. i.e., $x(x-4)(x+2)$

$$8x-8=A(x-4)(x+2)+Bx(x+2)+Cx(x-4)$$

(I)

Put $x=0$ in equation (I), we have

$$8(0)-8=A(0-4)(0+2)+B(0)(0+2)+C(0)(0-4)$$
$$-8=-8A+0+0$$
$$\Rightarrow \quad A=1$$

Put $x-4=0 \quad \Rightarrow \quad x=4$ in Equation (I), we have

$$8(4)-8=B(4)(4+2)$$
$$32-8=24B$$
$$24=24B$$
$$\Rightarrow \quad B=1$$

Put $x+2=0 \Rightarrow x=-2$ in Eq. (I), we have

$$8(-2)-8=C(-2)(-2-4)$$
$$-16-8=C(-2)(-6)$$
$$-24=12C$$
$$\Rightarrow \quad C=-2$$

Hence the required partial fractions

$$\frac{8x-8}{x^3-2x^2-8x}=\frac{1}{x}-\frac{1}{x-4}-\frac{2}{x+2}$$

Exercise 4.1

Resolve into partial fraction:

Q.1 $\dfrac{2x+3}{(x-2)(x+5)}$ Q.2 $\dfrac{2x+5}{x^2+5x+6}$

Q.3 $\dfrac{3x^2-2x-5}{(x-2)(x+2)(x+3)}$ Q.4 $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$

Q.5 $\dfrac{x}{(x-a)(x-b)(x-c)}$ Q.6 $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$

Q.7 $\dfrac{2x^3-x^2+1}{(x+3)(x-1)(x+5)}$ Q.8 $\dfrac{1}{(1-x)(1-2x)(1-3x)}$

Q.9 $\dfrac{6x+27}{4x^3-9x}$ Q.10 $\dfrac{9x^2-9x+6}{(x-1)(2x-1)(x+2)}$

Q.11 $\dfrac{x^4}{(x-1)(x-2)(x-3)}$ Q.12 $\dfrac{2x^3+x^2-x-3}{x(x-1)(2x+3)}$

Answers 4.1

Q.1 $\dfrac{1}{x-2}+\dfrac{1}{x+5}$ Q.2 $\dfrac{1}{x+2}+\dfrac{1}{x+3}$

Q.3 $\dfrac{3}{20(x-2)}-\dfrac{11}{4(x-2)}+\dfrac{28}{5(x+3)}$

Q.4 $1+\dfrac{3}{x-4}-\dfrac{24}{x-5}+\dfrac{30}{x-6}$

Q.5 $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-a)(b-c)(x-b)}+\dfrac{c}{(c-b)(c-a)(x-c)}$

Q.6 $\dfrac{a^2}{(a-b)(a-c)(1-ax)}+\dfrac{b^2}{(b-a)(b-c)(1-bx)}+\dfrac{c^2}{(c-b)(c-a)(1-cx)}$

Q.7 $2+\frac{31}{4(x+3)}+\frac{1}{12(x-1)}-\frac{137}{6(x+5)}$

Q.8 $\frac{1}{2(1-x)}-\frac{4}{(1-2x)}+\frac{9}{2(1-3x)}$

Q.9 $\frac{3}{x}+\frac{4}{2x-3}+\frac{2}{2x+3}$

Q.10 $\frac{2}{x-1}-\frac{3}{2x-1}+\frac{4}{x+12}$

Q.11 $x+6+\frac{1}{2(x-1)}-\frac{16}{x-2}+\frac{81}{2(x-3)}$

Q.12 $1+\frac{1}{x}-\frac{1}{5(x-1)}-\frac{8}{5(2x+3)}$

4.9 Type II:

When the factors of the denominator are all linear but some are repeated.

Example 1:

Resolve into partial fractions: $\frac{x^2-3x+1}{(x-1)^2(x-2)}$

Solution:

$$\frac{x^2-3x+1}{(x-1)^2(x-2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}$$

Multiplying both sides by L.C.M. i.e., $(x-1)^2(x-2)$, we get

$x^2-3x+1=A(x-1)(x-2)+B(x-2)+C(x-1)^2$ (1)

Putting $x-1=0$ $\Rightarrow$ $x=1$ in (1), then

$(1)^2-3(1)+1=B(1-2)$

$1-3+1=-B$

$-1=-B$

$\Rightarrow$ $B=1$

Putting $x-2=0$ $\Rightarrow$ $x=2$ in (1), then

$(2)^2-3(2)+1=C(2-1)^2$

$4-6+1=C(1)^2$

$\Rightarrow$ $-1=C$

Now $x^2-3x+1=A(x^2-3x+2)+B(x-2)+C(x^2-2x+1)$

Comparing the co-efficient of like powers of x on both sides, we get

$A+C=1$

$A=1-C$

$$= 1 - (-1)$$
$$= 1 + 1 = 2$$
$$\Rightarrow \quad A = 2$$

Hence the required partial fractions are

$$\frac{x^2 - 3x + 1}{(x-1)^2(x-2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x-2}$$

Example 2:

Resolve into partial fraction $\frac{1}{x^4(x+1)}$

Solution

$$\frac{1}{x^4(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+1}$$

Where A, B, C, D and E are constants. To find these constants multiplying both sides by L.C.M. i.e., $x^4\,(x + 1)$, we get

$$1 = A(x^3)(x + 1) + Bx^2\,(x + 1) + Cx\,(x + 1) + D(x + 1) + Ex^4$$

(I)

Putting $x = -1$ in Eq. (I)

$$1 = E(-1)^4$$
$$\Rightarrow \quad E = 1$$

Putting $x = 0$ in Eq. (I), we have

$$1 = D(0 + 1)$$
$$1 = D$$
$$\Rightarrow \quad D = 1$$

$$1 = A(x^4 + x^3) + B(x^3 + x^2) + C(x^2 + x) + D(x + 1) + Ex$$

Comparing the co-efficient of like powers of x on both sides.

Co-efficient of x^3 : $A + B = 0$ .... (i)

Co-efficient of x^2 : $B + C = 0$ .... (ii)

Co-efficient of x : $C + D = 0$ .... (iii)

Putting the value of $D = 1$ in (iii)

$$C + 1 = 0$$
$$\Rightarrow \quad C = -1$$

Putting this value in (ii), we get

$$B - 1 = 0$$
$$\Rightarrow \quad B = 1$$

Putting $B = 1$ in (i), we have

$$A + 1 = 0$$
$$\Rightarrow \quad A = -1$$

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Hence the required partial fraction are

$$\frac{1}{x^4(x+1)} = \frac{-1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x+1}$$

Example 3:

Resolve into partial fractions $\frac{4+7x}{(2+3x)(1+x)^2}$

Solution:

$$\frac{4+7x}{(2+3x)(1+x)^2} = \frac{A}{2+3x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$$

Multiplying both sides by L.C.M. i.e., $(2+3x)(1+x)^2$

We get $\quad 4+7x = A(1+x)^2 + B(2+3x)(1+x) + C(2+3x)$ .... (I)

Put $2+3x=0 \quad \Rightarrow \quad x = -\frac{2}{3}$ in (I)

Then $\quad 4+7\left(-\frac{2}{3}\right) = A\left(1-\frac{2}{3}\right)^2$

$$4-\frac{14}{3} = A\left(-\frac{1}{3}\right)^2$$

$$-\frac{2}{3} = \frac{1}{9}A$$

$$\Rightarrow \quad A = \frac{-2}{3} x \frac{9}{1} = -6$$

$$A = -6$$

Put $\quad 1+x=0 \quad \Rightarrow \quad x=-1$ in eq. (I), we get

$$4+7(-1) = C(2-3)$$

$$4-7 = C(-1)$$

$$-3 = -C$$

$$\Rightarrow \quad C = 3$$

$$4+7x = A(x^2+2x+1) + B(2+5x+3x^2) + C(2+3x)$$

Comparing the co-efficient of x^2 on both sides

$$A + 3B = 0$$

$$-6 + 3B = 0$$

$$3B = 6$$

$$\Rightarrow \quad B = 2$$

Hence the required partial fraction will be

$$\frac{-6}{2+3x} + \frac{2}{1+x} + \frac{3}{(1+x)^2}$$

Exercise 4.2

Resolve into partial fraction:

Q.1 $\dfrac{x+4}{(x-2)^2(x+1)}$

Q2. $\dfrac{1}{(x+1)(x^2-1)}$

Q.3 $\dfrac{4x^3}{(x+1)^2(x^2-1)}$

Q.4 $\dfrac{2x+1}{(x+3)(x-1)(x+2)^2}$

Q.5 $\dfrac{6x^2-11x-32}{(x+6)(x+1)^2}$

Q.6 $\dfrac{x^2-x-3}{(x-1)^3}$

Q.7 $\dfrac{5x^2+36x-27}{x^4-6x^3+9x^2}$

Q.8 $\dfrac{4x^2-13x}{(x+3)(x-2)^2}$

Q.9 $\dfrac{x^4+1}{x^2(x-1)}$

Q.10 $\dfrac{x^3-8x^2+17x+1}{(x-3)^3}$

Q.11 $\dfrac{x^2}{(x-1)^3(x+2)}$

Q.12 $\dfrac{2x+1}{(x+2)(x-3)^2}$

Answers4.2

Q.1 $-\dfrac{1}{3(x-2)}+\dfrac{2}{(x-2)^2}+\dfrac{1}{3(x+1)}$

Q.2 $\dfrac{1}{4(x-1)}-\dfrac{1}{4(x+1)}-\dfrac{1}{2(x+1)^2}$

Q.3 $\dfrac{1}{2(x-1)}+\dfrac{7}{2(x+1)}-\dfrac{5}{(x+1)^2}+\dfrac{2}{(x+1)^3}$

Q.4 $\dfrac{5}{4(x+3)}+\dfrac{1}{12(x-1)}-\dfrac{4}{3(x+2)}+\dfrac{1}{(x+2)^2}$

Q.5 $\dfrac{10}{x+6}-\dfrac{4}{x+1}-\dfrac{3}{(x-1)^2}$

Q.6 $\dfrac{1}{x-1}+\dfrac{1}{(x-1)^2}-\dfrac{3}{(x-1)^3}$

Q.7 $\dfrac{2}{x}-\dfrac{3}{x^2}-\dfrac{2}{(x-3)}+\dfrac{14}{(x-3)^2}$

Q.8 $\dfrac{3}{x+3}+\dfrac{1}{x-2}-\dfrac{2}{(x-2)^2}$

Q.9 $x+1-\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x-1}$

Q.10 $1+\frac{1}{x-3}-\frac{4}{(x-3)^2}+\frac{7}{(x-3)^3}$

Q.11 $\frac{4}{27(x-1)}+\frac{5}{9(x-1)^2}+\frac{1}{3(x-1)^3}-\frac{4}{27(x+2)}$

Q.12 $-\frac{3}{25(x+2)}+\frac{3}{25(x-3)}+\frac{7}{5(x-3)^2}$

4.10 Type III:

When the denominator contains ir-reducible quadratic factors which are non-repeated.

Example 1:

Resolve into partial fractions $\frac{9x-7}{(x+3)(x^2+1)}$

Solution:

$$\frac{9x-7}{(x+3)(x^2+1)}=\frac{A}{x+3}+\frac{Bx+C}{x^2+1}$$

Multiplying both sides by L.C.M. i.e., $(x+3)(x^2+1)$, we get

$$9x-7=A(x^2+1)+(Bx+C)(x+3) \qquad \text{(I)}$$

Put $x+3=0 \Rightarrow x=-3$ in Eq. (I), we have

$$9(-3)-7=A((-3)^2+1)+(B(-3)+C)(-3+3)$$

$$-27-7=10A+0$$

$$A=-\frac{34}{10} \Rightarrow \boxed{A=-\frac{17}{5}}$$

$$9x-7=A(x^2+1)+B(x^2+3x)+C(x+3)$$

Comparing the co-efficient of like powers of x on both sides

$$A+B=0$$

$$3B+C=9$$

Putting value of A in Eq. (i)

$$-\frac{17}{5}+B=0 \Rightarrow \boxed{B=\frac{17}{5}}$$

From Eq. (iii)

$$C=9-3B=9-3\left(\frac{17}{4}\right)$$

$$=9-\frac{51}{5} \Rightarrow \boxed{C=-\frac{6}{5}}$$

Hence the required partial fraction are

$$\frac{-17}{5(x+3)}+\frac{17x-6}{5(x^2+1)}$$

Example 2:

Resolve into partial fraction $\dfrac{x^2+1}{x^4+x^2+1}$

Solution:

Let $$\frac{x^2+1}{x^4+x^2+1}=\frac{x^2+1}{(x^2-x+1)(x^2+x+1)}$$

$$\frac{x^2+1}{(x^2-x+1)(x^2+x+1)}=\frac{Ax+B}{(x^2-x+1)}+\frac{Cx+D}{(x^2+x+1)}$$

Multiplying both sides by L.C.M. i.e., $(x^2-x+1)(x^2+x+1)$

$x^2+1=(Ax+B)(x^2+x+1)+(Cx+D)(x^2-x+1)$

Comparing the co-efficient of like powers of x, we have

Co-efficient of x^3 : $A+C=0$ .... (i)

Co-efficient of x^2 : $A+B-C+D=1$ .... (ii)

Co-efficient of x : $A+B+C-D=0$ .... (iii)

Constant $B+D=1$ .... (iv)

Subtract (iv) from (ii) we have

$A-C=0$ .... (v)

$A=C$ .... (vi)

Adding (i) and (v), we have

$A=0$

Putting $A=0$ in (vi), we have

$C=0$

Putting the value of A and C in (iii), we have .... (vii)

$B-D=0$

Adding (iv) and (vii)

$2B=1 \quad \Rightarrow \quad B=\dfrac{1}{2}$

from (vii) $B=D$, therefore

$D=\dfrac{1}{2}$

Hence the required partial fraction are

$$\frac{0x+\frac{1}{2}}{(x^2-x+1)}+\frac{0x+\frac{1}{2}}{(x^2+x+1)}$$

i.e., $$\frac{1}{2(x^2-x+1)}+\frac{1}{2(x^2+x+1)}$$

Exercise 4.3

Resolve into partial fraction:

Q.1 $\dfrac{x^2+3x-1}{(x-2)(x^2+5)}$ Q.2 $\dfrac{x^2-x+2}{(x+1)(x^2+3)}$

Q.3 $\dfrac{3x+7}{(x+3)(x^2+1)}$ Q.4 $\dfrac{1}{(x^3+1)}$

Q.5 $\dfrac{1}{(x+1)(x^2+1)}$ Q.6 $\dfrac{3x+7}{(x^2+x+1)(x^2-4)}$

Q.7 $\dfrac{3x^2-x+1}{(x+1)(x^2-x+3)}$ Q.8 $\dfrac{x+a}{x^2(x-a)(x^2+a^2)}$

Q.9 $\dfrac{x^5}{x^4-1}$ Q.10 $\dfrac{x^2+x+1}{(x^2-x-2)(x^2-2)}$

Q.11 $\dfrac{1}{x^3-1}$ Q.12 $\dfrac{x^2+3x+3}{(x^2-1)(x^2+4)}$

Answers 4.3

Q.1 $\dfrac{1}{x-2}+\dfrac{3}{x^2+5}$ Q.2 $\dfrac{1}{x+1}-\dfrac{1}{x^2+3}$

Q.3 $-\dfrac{1}{5(x+3)}+\dfrac{x+12}{5(x^2+1)}$ Q.4 $\dfrac{1}{3(x+1)}-\dfrac{(x-2)}{3(x^2-x+1)}$

Q.5 $\dfrac{1}{2(x+1)}-\dfrac{x-1}{2(x^2+1)}$

Q.6 $\dfrac{13}{28(X-2)}-\dfrac{1}{12(X+2)}-\dfrac{8X+31}{21(X^2+X+1)}$

Q.7 $\dfrac{1}{x+1}+\dfrac{2x-2}{x^2-x+3}$

Q.8 $\dfrac{1}{a^3}\left[\dfrac{1}{X-a}+\dfrac{x}{X^2+a^2}-\dfrac{2}{X}-\dfrac{a}{X^2}\right]$

Q.9 $x+\frac{1}{4(x-1)}+\frac{1}{4(x+1)}-\frac{x}{2(x^2+1)}$

Q.10 $\frac{1}{3(x+1)}+\frac{7}{6(x-2)}-\frac{3x+2}{2(x^2-2)}$

Q.11 $\frac{1}{3(x-1)}-\frac{x+2}{3(x^2+x+1)}$

Q.12 $\frac{7}{10(x-1)}-\frac{1}{10(x+1)}-\frac{3x-1}{5(x^2+4)}$

4.11 Type IV: Quadratic repeated factors

When the denominator has repeated Quadratic factors.

Example 1:

Resolve into partial fraction $\frac{x^2}{(1-x)(1+x^2)^2}$

Solution:

$$\frac{x^2}{(1-x)(1+x^2)^2}=\frac{A}{1-x}+\frac{Bx+C}{(1+x^2)}+\frac{Dx+E}{(1+x^2)^2}$$

Multiplying both sides by L.C.M. i.e., $(1-x)(1+x^2)^2$ on both sides, we have

$x^2 = A(1+x^2)^2+(Bx+C)(1-x)(1+x^2)+(Dx+E)(1-x)$ ......(i)

$x^2 = A(1+2x^2+x^4)+(Bx+C)(1-x+x^2-x^3)+(Dx+E)(1-x)$

Put $1-x=0 \Rightarrow$ $x=1$ in eq. (i), we have

$(1)^2 = A(1+(1)^2)^2$

$1=4A \Rightarrow \boxed{A=\frac{1}{4}}$

$x^2 = A(1+2x^2+x^4)+B(x-x^2+x^3-x^4)+C(1-x+x^2-x^3)$ .... (ii)

$+D(x-x^2)+E(1-x)$

Comparing the co-efficients of like powers of x on both sides in Equation (II), we have

Co-efficient of x^4 : $A-B=0$ .... (i)

Co-efficient of x^3 : $B-C=0$ .... (ii)

Co-efficient of x^2 : $2A-B+C-D=1$ .... (iii)

Co-efficient of x : $B-C+D-E=0$ .... (iv)

Co-efficient term : $A+C+E=0$ .... (v)

from (i), $B=A$

$\Rightarrow B=\frac{1}{4}$ $\because A=\frac{1}{4}$

hs

from (i)

$$B = C$$

$$\Rightarrow \quad C = \frac{1}{4}$$

from (iii) $\quad \because \quad C = \frac{1}{4}$

$$D = 2A - B + C - 1$$

$$= 2\left(\frac{1}{4}\right) - \frac{1}{4} + \frac{1}{4} - 1$$

$$\Rightarrow \quad \boxed{D = -\frac{1}{2}}$$

from (v)

$$E = -A - C$$

$$E = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

Hence the required partial fractions are by putting the values of A, B, C, D, E,

$$\frac{\frac{1}{4}}{1-x} + \frac{\frac{1}{4}x + \frac{1}{4}}{1+x^2} + \frac{-\frac{1}{2}x - \frac{1}{2}}{(1+x^2)^2}$$

$$\frac{1}{4(1-x)} + \frac{(x+1)}{4(1+x^2)} - \frac{x+1}{2(1+x^2)^2}$$

Example 2:

Resolve into partial fractions $\frac{x^2+x+2}{x^2(x^2+3)^2}$

Solution:

Let $$\frac{x^2+x+2}{x^2(x^2+3)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+3} + \frac{Ex+F}{(x^2+3)^2}$$

Multiplying both sides by L.C.M. i.e., $x^2(x^2+3)^2$, we have

$$x^2+x+2 = Ax(x^2+3)^2 + B(x^2+3)^2$$
$$+(cx+D)x^2(x^2+3) + (Ex+F)(x^2)$$

Putting $x = 0$ on both sides, we have

$$2 = B(0+3)^2$$

$$2 = 9B \quad \Rightarrow \quad \boxed{B = \frac{2}{9}}$$

Now $$x^2+x+2 = Ax(x^4+6x^2+9) + B(x^4+6x^2+9)$$
$$+C(x^5+3x^2) + D(x^4+3x^2) + E(x^3) + Fx^2$$

$$x^2+x+2 = (A+C)x^5 + (B+D)x^4 + (6A+3C+E)x^3$$

$$+(6B+3D+F)x^2 + (x+9B)$$

Comparing the co-efficient of like powers of x on both sides of Eq. (I), we have

Co-efficient of x^5 : $A + C = 0$ .... (i)

Co-efficient of x^4 : $B - D = 0$ .... (ii)

Co-efficient of x^3 : $6A + 3C + E = 0$ .... (iii)

Co-efficient of x^2 : $6B + 3D + F = 1$ .... (iv)

Co-efficient of x : $9A = 1$ .... (v)

Co-efficient term : $9B = 1$ .... (vi)

from (v) $9A = 1$

$\Rightarrow$ $\boxed{A = \frac{1}{9}}$

from (i) $A + C = 0$

$C = -A$

$\Rightarrow$ $\boxed{C = -\frac{1}{9}}$

from (i) $B + D = 0$

$D = -B$

$\Rightarrow$ $\boxed{D = -\frac{2}{9}}$

from (iii) $6A + 3C + E =$

$$6\left(\frac{1}{9}\right) + 3\left(-\frac{1}{9}\right) + E = 0$$

$$E = \frac{3}{9} - \frac{6}{9}$$

$\Rightarrow$ $\boxed{E = -\frac{1}{3}}$

from (iv) $6B + 3D + F = 1$

$$F = 1 - 6B - 3D$$

$$= 1 - 6\left(\frac{2}{9}\right) - 3\left(\frac{2}{9}\right)$$

$$=1-\frac{12}{9}+\frac{6}{9}$$

$$\Rightarrow \boxed{F=\frac{1}{3}}$$

Hence the required partial fractions are

$$\frac{\frac{1}{9}}{x}+\frac{\frac{2}{9}}{x^2}+\frac{-\frac{1}{9}x-\frac{2}{9}}{x^2+3}+\frac{-\frac{1}{3}x+\frac{1}{3}}{(x^2+3)^2}$$

$$=\frac{1}{9x}+\frac{2}{9x^2}-\frac{x+2}{9(x^2+3)}-\frac{x-1}{3(x^2+3)^2}$$

Exercise 4.4

Resolve into Partial Fraction:

Q.1 $\dfrac{7}{(x+1)(x^2+2)^2}$

Q.2 $\dfrac{x^2}{(1+x)(1+x^2)^2}$

Q.3 $\dfrac{5x^2+3x+9}{x(x^2+3)^2}$

Q.4 $\dfrac{4x^4+3x^3+6x^2+5x}{(x-1)(x^2+x+1)^2}$

Q.5 $\dfrac{2x^4-3x^2-4x}{(x+1)(x^2+2)^2}$

Q.6 $\dfrac{x^3-15x^2-8x-7}{(2x-5)(1+x^2)^2}$

Q.7 $\dfrac{49}{(x-2)(x^2+3)^2}$

Q.8 $\dfrac{8x^2}{(1-x^2)(1+x^2)^2}$

Q.9 $\dfrac{x^4+x^3+2x^2-7}{(x+2)(x^2+x+1)^2}$

Q.10 $\dfrac{x^2+2}{(x^2+1)(x^2+4)^2}$

Q.11 $\dfrac{1}{x^4+x^2+1}$

Answers 4.4

Q.1 $\dfrac{7}{9(X+1)}-\dfrac{7X-7}{9(X^2+2)}-\dfrac{7X-7}{3(X^2+2)^2}$

Q.2 $\dfrac{1}{4(1+x)}-\dfrac{x-1}{4(1+x^2)}+\dfrac{x-1}{2(1+x^2)^2}$

Q.3 $\dfrac{1}{x}-\dfrac{x}{x^2+3}+\dfrac{2x+3}{(x^2+3)^2}$

Q.4 $\dfrac{2}{X-1}+\dfrac{2X-1}{X^2+X+1}+\dfrac{3}{(X^2+X+1)^2}$

Q.5 $\frac{1}{3(x+1)}+\frac{5(x-1)}{3(x^2+2)}-\frac{2(3x-1)}{(x^2+1)^2}$

Q.6 $-\frac{2}{2x-5}+\frac{x+3}{1+x^2}+\frac{x-2}{(1+x^2)^2}$

Q.7 $\frac{1}{x-2}-\frac{x+2}{x^2+3}-\frac{7x+14}{(x^2+3)^2}$

Q.8 $\frac{1}{1-x}+\frac{1}{1+x}+\frac{2}{1+x^2}-\frac{4}{(1+x^2)^2}$

Q.9 $\frac{1}{x+2}+\frac{2x-3}{(x^2+x+1)^2}-\frac{1}{x^2+x+1}$

Q.10 $\frac{1}{9(x^2+1)}-\frac{1}{9(x^2+4)}+\frac{2}{3(x^2+4)^2}$

Q.11 $-\frac{(x-1)}{2(x^2-x+1)}+\frac{(x+1)}{2(x^2+x+1)}$

Summary

Let $N(x) \neq$ and $D(x) \neq 0$ be two polynomials. The $\frac{N(x)}{D(x)}$ is called a proper fraction if the degree of N(x) is smaller than the degree of D(x).

For example: $\frac{x-1}{x^2+5x+6}$ is a proper fraction.

Also $\frac{N(x^1)}{D(x)}$ is called an improper fraction of the degree of N(x) is greater than or equal to the degree of D(x).

For example: $\frac{x^5}{x^4-1}$ is an improper fraction.

In such problems we divide N(x) by D(x) obtaining a quotient Q(x) and a remainder R(x) whose degree is smaller than that of D(x).

Thus $\frac{N(x)'}{D(x)}=Q(x)+\frac{R(x)}{D(x)}$ where $\frac{R(x)'}{D(x)}$ is proper fraction.

Types of proper fraction into partial fractions.

Type 1:

Linear and distinct factors in the D(x)

$$\frac{x-a}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

Type 2:

Linear repeated factors in D(x)

$$\frac{x-a}{(x+a)(x^2+b^2)} = \frac{A}{x+a} + \frac{Bx+C}{x^2+b^2}$$

Type 3:

Quadratic Factors in the D(x)

$$\frac{x-a}{(x+a)(x^2+b)^2} = \frac{A}{x+a} + \frac{Bx+C}{x^2+b^2}$$

Type 4:

Quadratic repeated factors in D(x):

$$\frac{x-a}{(x^2+a^2)(x^2+b^2)} = \frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{x^2+b^2} + \frac{Ex+F}{(x^2+b^2)^2}$$

Short Questions:

Write the short answers of the following:

Q.1: What is partial fractions?

Q.2: Define proper fraction and give example.

Q.3: Define improper fraction and given one example:

Q.4: Resolve into partial fractions $\frac{2x}{(x - 2)(x + 5)}$

Q.5: Resolve into partial fractions: $\frac{1}{x^2 - x}$

Q.6: Resolve $\frac{7x + 25}{(x + 3)(x + 4)}$ into partial fraction.

Q.7: Resolve $\frac{1}{x^2 - 1}$ into partial fraction:

Q.8: Resolve $\frac{x^2 + 1}{(x + 1)(x - 1)}$ into partial fractions.

Q.9: Write an identity equation of $\frac{8x^2}{(1 - x^2)(1 + x^2)^2}$

Q.10: Write an identity equation of $\frac{2x + 5}{x^2 + 5x + 6}$

Q.11: Write identity equation of $\frac{x - 5}{(x + 1)(x^2 + 3)}$

Q.12: Write an identity equation of $\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1}$

Q.13: Write an identity equation of $\frac{(x - 1)(x - 2)(x - 3)}{(x - 4)(x - 5)(x - 6)}$

Q.14: Write an identity equation of $\frac{x^5}{x^4 - 1}$

Q.15: Write an identity equation of $\frac{2x^4 - 3x^2 - 4x}{(x + 1)(x^2 + 2)^2}$

Q16. Form of partial fraction of $\dfrac{1}{(x+1)(x-2)}$ is ________.

Q.17. Form of partial fraction of $\dfrac{1}{(x+1)^2(x-2)}$ is ________.

Q.18. Form of partial fraction of $\dfrac{1}{(x^2+1)(x-2)}$ is ________.

Q.19. Form of partial fraction of $\dfrac{1}{(x^2+1)(x-4)^2}$ is ________.

Q.20. Form of partial fraction of $\dfrac{1}{(x^3-1)(x^2+1)}$ is ________.

Answers

Q4. $\dfrac{4}{7(x-2)} - \dfrac{10}{7(x+5)}$

Q5. $\dfrac{-1}{x} + \dfrac{1}{x-1}$

Q6. $\dfrac{4}{x+3} + \dfrac{3}{x+4}$

Q7. $\dfrac{1}{x^2-1} = \dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)}$

Q8. $1 + \dfrac{1}{x+1} + \dfrac{1}{x-1}$

Q9. $\dfrac{A}{1-x} + \dfrac{B}{1+x} + \dfrac{Cx+D}{1+x^2} + \dfrac{Ex+F}{(1+x^2)^2}$

Q10. $\dfrac{A}{x+2} + \dfrac{B}{x+3}$

Q11. $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+3}$

Q12. $(2x+3) + \dfrac{A}{x-1} + \dfrac{B}{3x+1}$

Q13. $1 + \dfrac{A}{4-4} + \dfrac{B}{x-5} + \dfrac{C}{x-6}$

Q14. $x + \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1}$

Q15. $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+2} + \dfrac{Dx+E}{(x^2+2)^2}$

Q16. $\dfrac{A}{x+1} + \dfrac{B}{x-2}$

Q17. $\dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-2}$

Q18. $\dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-2}$

Q19. $\dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2}$

Q20. $\dfrac{A}{(x-1)} + \dfrac{Bx+C}{(x^2+x+1)} + \dfrac{Dx+E}{x^2+1}$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. If the degree of numerator N(x) is equal or greater than the degree of denominator D(x), then the fraction is:
(a) proper
(b) improper
(c) Neither proper non-improper
(d) Both proper and improper

__2. If the degree of numerator is less than the degree of denominator, then the fraction is:
(a) Proper
(b) Improper
(c) Neither proper non-improper
(d) Both proper and improper

__3. The fraction $\frac{2x+5}{x^2+5x+6}$ is known as:
(a) Proper
(b) Improper
(c) Both proper and improper
(d) None of these

__4. The number of partial fractions of $\frac{6x+27}{4x^3-9x}$ are:
(a) 2
(b) 3
(c) 4
(d) None of these

__5. The number of partial fractions of $\frac{x^3-3x^2+1}{(x-1)(x+1)(x^2-1)}$ are:
(a) 2
(b) 3
(c) 4
(d) 5

__6. The equivalent partial fraction of $\frac{x+11}{(x+1)(x-3)^2}$ is:
(a) $\frac{A}{x+1}+\frac{B}{(x-3)^2}$
(b) $\frac{A}{x+1}+\frac{B}{x-3}$
(c) $\frac{A}{x+1}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$
(d) $\frac{A}{x+1}+\frac{Bx+C}{(x-3)^2}$

__7. The equivalent partial fraction of $\frac{x^4}{(x^2+1)(x^2+3)}$ is:
(a) $\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+3}$
(b) $\frac{Ax+B}{x^2+1}+\frac{Cx}{x^2+3}$
(c) $1+\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+3}$
(d) $\frac{Ax}{x^2+1}+\frac{Bx}{x^2+3}$

_8. Partial fraction of $\dfrac{2}{x(x+1)}$ is:

(a) $\dfrac{2}{x}-\dfrac{1}{x+1}$ (b) $\dfrac{1}{x}-\dfrac{2}{x+1}$

(c) $\dfrac{2}{x}-\dfrac{2}{x+1}$ (d) $\dfrac{2}{x}+\dfrac{2}{x+1}$

_9. Partial fraction of $\dfrac{2x+3}{(x-2)(x+5)}$ is called:

(a) $\dfrac{2}{x-2}+\dfrac{1}{x+5}$ (b) $\dfrac{3}{x-2}+\dfrac{1}{x+5}$

(c) $\dfrac{2}{x-2}+\dfrac{3}{x+5}$ (d) $\dfrac{1}{x-2}+\dfrac{1}{x+5}$

_10. The fraction $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$ is called:

(a) Proper (ii) Improper

(c) Both proper and Improper (iv) None of these

Answers:

1.	b	2.	a	3.	a	4.	b	5.	c
6.	c	7.	c	8.	c	9.	d	10.	B

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DEFINITE INTEGRALS

In the previous lesson we have discussed the anti-derivative, i.e., integration of a function. The very word integration means to have some sort of summation or combining of results.

Now the question arises : Why do we study this branch of Mathematics? In fact the integration helps to find the areas under various laminas when we have definite limits of it. Further we will see that this branch finds applications in a variety of other problems in Statistics, Physics, Biology, Commerce and many more.

In this lesson, we will define and interpret definite integrals geometrically, evaluate definite integrals using properties and apply definite integrals to find area of a bounded region.

OBJECTIVES

After studying this lesson, you will be able to :

- define and interpret geometrically the definite integral as a limit of sum;
- evaluate a given definite integral using above definition;
- state fundamental theorem of integral calculus;
- state and use the following properties for evaluating definite integrals :

(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(ii) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

(iii) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

(iv) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(v) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

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Notes

(vi) $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$

(vii) $\int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ if f is an even function of x

$= 0$ if f is an odd function of x.

- apply definite integrals to find the area of a bounded region.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of integration
- Area of a bounded region

27.1 DEFINITE INTEGRAL AS A LIMIT OF SUM

In this section we shall discuss the problem of finding the areas of regions whose boundary is not familiar to us. (See Fig. 27.1)

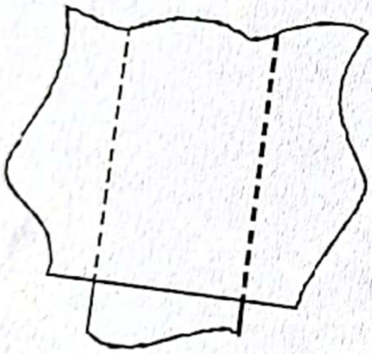


Fig. 27.1

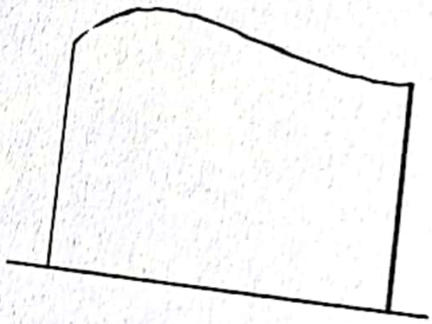
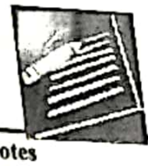


Fig. 27.2

Let us restrict our attention to finding the areas of such regions where the boundary is not familiar to us is on one side of x-axis only as in Fig. 27.2.

This is because we expect that it is possible to divide any region into a few subregions of this kind, find the areas of these subregions and finally add up all these areas to get the area of the whole region. (See Fig. 27.1)

Now, let $f(x)$ be a continuous function defined on the closed interval $[a, b]$. For the present, assume that all the values taken by the function are non-negative, so that the graph of the function is a curve above the x-axis (See. Fig.27.3).



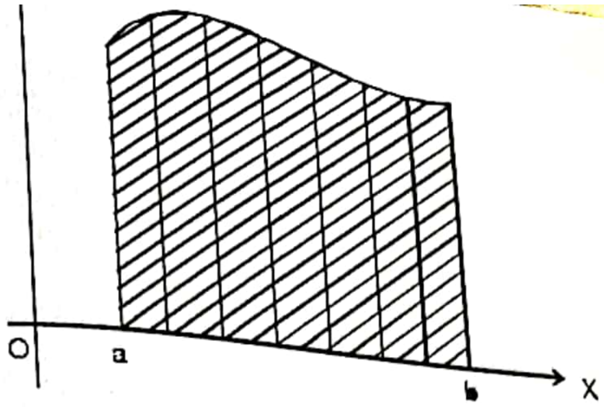


Fig. 27.3

Consider the region between this curve, the x-axis and the ordinates $x = a$ and $x = b$, that is, the shaded region in Fig.27.3. Now the problem is to find the area of the shaded region.

In order to solve this problem, we consider three special cases of $f(x)$ as rectangular region, triangular region and trapezoidal region.

The area of these regions $=$ base $\times$ average height

In general for any function $f(x)$ on $[a, b]$

Area of the bounded region (shaded region in Fig. 27.3) $=$ base $\times$ average height

The base is the length of the domain interval [a, b]. The height at any point x is the value of $f(x)$ at that point. Therefore, the average height is the average of the values taken by f in [a, b]. (This may not be so easy to find because the height may not vary uniformly.) Our problem is how to find the average value of f in [a,b].

27.1.1 Average Value of a Function in an Interval

If there are only finite number of values of f in [a,b], we can easily get the average value by the formula.

$$\text{Average value of f in } [a, b] = \frac{\text{Sumof thevaluesof f in } [a, b]}{\text{Numbersof values}}$$

But in our problem, there are infinite number of values taken by f in [a, b]. How to find the average in such a case? The above formula does not help us, so we resort to estimate the average value of f in the following way:

First Estimate : Take the value of f at 'a' only. The value of f at a is f (a). We take this value, namely f (a), as a rough estimate of the average value of f in [a,b].

Average value of f in [a, b] (first estimate) $= f(a)$ (i)

Second Estimate : Divide [a, b] into two equal parts or sub-intervals.

Let the length of each sub-interval be h, $h = \dfrac{b-a}{2}$.

Take the values of f at the left end points of the sub-intervals. The values are $f(a)$ and $f(a+h)$

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(Fig. 27.4)

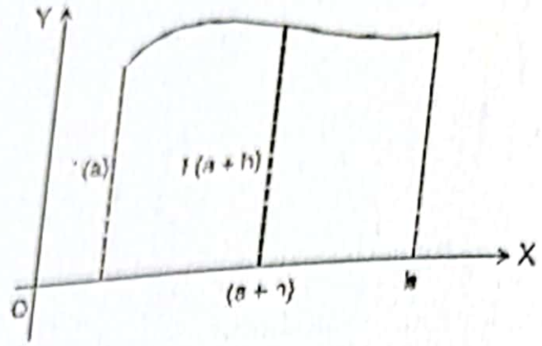

Fig. 27.4

Take the average of these two values as the average of f in [a, b].

Average value of f in [a, b] (Second estimate)

$$= \frac{f(a)+f(a+h)}{2}, \quad h = \frac{b-a}{2} \qquad \text{(ii)}$$

This estimate is expected to be a better estimate than the first.

Proceeding in a similar manner, divide the interval [a, b] into n subintervals of length h (Fig. 27.5), $h = \frac{b-a}{n}$

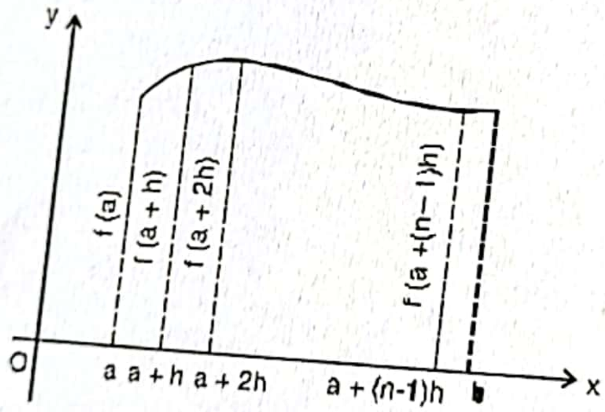

Fig. 27.5

Take the values of f at the left end points of the n subintervals.

The values are f (a), f (a + h),......,f [a + (n-1) h]. Take the average of these n values of f in [a, b].

Average value of f in [a, b] (nth estimate)

$$= \frac{f(a)+f(a+h)+\ldots\ldots\ldots+f(a+(n-1)h)}{n}, \quad h = \frac{b-a}{n} \qquad \text{(iii)}$$

For larger values of n, (iii) is expected to be a better estimate of what we seek as the average value of f in [a, b]

Thus, we get the following sequence of estimates for the average value of f in [a, b]:

$f(a)$

$\frac{1}{2}[f(a)+f(a+h)]$, $\qquad h=\frac{b-a}{2}$

$\frac{1}{3}[f(a)+f(a+h)+f(a+2h)]$, $\qquad h=\frac{b-a}{3}$

....

....

$\frac{1}{n}[f(a)+f(a+h)+\ldots\ldots+f\{a+(n-1)h\}]$, $h=\frac{b-a}{n}$

As we go farther and farther along this sequence, we are going closer and closer to our destination, namely, the average value taken by f in [a, b]. Therefore, it is reasonable to take the limit of these estimates as the average value taken by f in [a, b]. In other words,

Average value of f in [a, b]

$$\lim_{n\to\infty}\frac{1}{n}\{f(a)+f(a+h)+f(a+2h)+\ldots+f[a+(n-1)h]\},$$

$$h=\frac{b-a}{n} \qquad \text{(iv)}$$

It can be proved that this limit exists for all continuous functions f on a closed interval [a, b]. Now, we have the formula to find the area of the shaded region in Fig. 27.3, The base is $(b-a)$ and the average height is given by (iv). The area of the region bounded by the curve f (x), x-axis, the ordinates $x=a$ and $x=b$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\{f(a)+f(a+h)+f(a+2h)+\ldots+f[a+(n-1)h]\},$$

$$\lim_{n\to 0}\frac{1}{n}[f(a)+f(a+h)+\ldots\ldots+f\{a+(n-1)h\}],\ h=\frac{b-a}{n} \qquad \text{(v)}$$

We take the expression on R.H.S. of (v) as the definition of a **definite integral.** This integral is denoted by

$$\int_a^b f(x)\,dx$$

read as integral of f (x) from a to b'. The numbers a and b in the symbol $\int_a^b f(x)\,dx$ are called respectively the lower and upper limits of integration, and f (x) is called the integrand.

Note : In obtaining the estimates of the average values of f in [a, b], we have taken the left end points of the subintervals. Why left end points?

subintervals throughout and in that case we get

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\{f(a+h)+f(a+2h)+\dots+f(b)\},\quad h=\frac{b-a}{n}$$

$$=\lim_{h\to 0} h\,[f(a+h)+f(a+2h)+\dots\dots+f(b)] \qquad \text{(vi)}$$

Example 27.1 Find $\int_1^2 x\,dx$ as the limit of sum.

Solution : By definition,

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+\dots\dots+f\{a+(n-1)h\}],$$

$$h=\frac{b-a}{n}$$

Here $a=1$, $b=2$, $f(x)=x$ and $h=\frac{1}{n}$.

$$\therefore \quad \int_1^2 x\,dx = \lim_{n\to\infty}\frac{1}{n}\left[f(1)+f\left(1+\frac{1}{n}\right)+\dots\dots+f\left(1+\frac{n-1}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{1}{n}\right)+\left(1+\frac{2}{n}\right)+\dots\dots+\left(1+\frac{n-1}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\underbrace{1+1+\dots\dots+1}_{n\,\text{times}}+\left(\frac{1}{n}+\frac{2}{n}+\dots+\frac{n-1}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{1}{n}(1+2+\dots\dots+(n-1))\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{(n-1).n}{n.2}\right]$$

$$\left[\text{Since } 1+2+3+\dots+(n-1)=\frac{(n-1).n}{2}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{3n-1}{2}\right]$$

$$=\lim_{n\to\infty}\left[\frac{3}{2}-\frac{1}{2n}\right]=\frac{3}{2}$$

Example 27.2 Find $\int_0^2 e^x\,dx$ as limit of sum.

Solutions : By definition

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f\{a+(n-1)h\}\right]$$

where

$$h = \frac{b-a}{n}$$

Here $a = 0, b = 2, f(x) = e^x$ and $h = \frac{2-0}{n} = \frac{2}{n}$

$$\therefore \int_0^2 e^x\,dx = \lim_{h\to 0} h\left[f(0) + f(h) + f(2h) + \ldots + f(n-1)h\right]$$

$$= \lim_{h\to 0} h\left[e^0 + e^h + e^{2h} + \ldots + e^{(n-1)h}\right]$$

$$= \lim_{h\to 0} h\left[e^0\left(\frac{\left(e^h\right)^n - 1}{e^h - 1}\right)\right]$$

$$\left[\text{Since } a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\frac{r^n-1}{r-1}\right)\right]$$

$$= \lim_{h\to 0} h\left[\frac{e^{nh}-1}{e^h-1}\right] = \lim_{h\to 0}\frac{h}{h}\left[\frac{e^2-1}{\left(\frac{e^h-1}{h}\right)}\right] \qquad (\because nh = 2)$$

$$= \lim_{h\to 0}\frac{e^2-1}{\frac{e^h-1}{h}} = \frac{e^2-1}{1}$$

$$\left[\because \lim_{h\to 0}\frac{e^h-1}{h} = 1\right]$$

$$= e^2 - 1$$

In examples 27.1 and 27.2 we observe that finding the definite integral as the limit of sum is quite difficult. In order to overcome this difficulty we have the fundamental theorem of integral calculus which states that

Theorem 1 : If f is continuous in $[a, b]$ and F is an antiderivative of f in [a, b] then

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \ldots\ldots(1)$$

The difference F (b)−F (a) is commonly denoted by $\left[F(x)\right]_a^b$ so that (1) can be written as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \text{ or } \left[F(x)\right]_a^b$$

In words, the theorem tells us that

$$\int_a^b f(x)\,dx = \text{(Value of antiderivative at the upper limit b)}$$

$$-\text{(Value of the same antiderivative at the lower limit a)}$$

Example 27.3 Find $\int_1^2 x\,dx$

Solution : $$\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

Example 27.4 Evaluate the following

(a) $\int_0^{\frac{\pi}{2}} \cos x\,dx$ (b) $\int_0^2 e^{2x}\,dx$

Solution : We know that

$$\int \cos x dx = \sin x + c$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \cos x dx = [\sin x]_0^{\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

(b) $$\int_0^2 e^{2x}\,dx = \left[\frac{e^{2x}}{2}\right]_0^2, \qquad \left[\because \int e^x dx = e^x\right]$$

$$= \left(\frac{e^4 - 1}{2}\right)$$

Theorem 2 : If f and g are continuous functions defined in [a, b] and c is a constant then,

(i) $$\int_a^b c f(x) dx = c\int_a^b f(x) dx$$

(ii) $$\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx \quad \int_a^b g(x) dx$$

(iii) $$\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

Example 27.5 Evaluate $\int_0^2 \left(4x^2 - 5x + 7\right) dx$

Solution : $\int_0^2 \left(4x^2 - 5x + 7\right) dx = \int_0^2 4x^2 dx - \int_0^2 5x\, dx + \int_0^2 7\, dx$

$$= 4\int_0^2 x^2 dx - 5\int_0^2 x dx + 7\int_0^2 1\, dx$$

$$= 4.\left[\frac{x^3}{3}\right]_0^2 - 5\left[\frac{x^2}{2}\right]_0^2 + 7[x]_0^2$$

$$= 4.\left(\frac{8}{3}\right) - 5\left(\frac{4}{2}\right) + 7(2)$$

$$= \frac{32}{3} - 10 + 14$$

$$= \frac{44}{3}$$

CHECK YOUR PROGRESS 27.1

1. Find $\int_0^5 (x+1)\, dx$ as the limit of sum.
2. Find $\int_{-1}^1 e^x\, dx$ as the limit of sum.
3. Evaluate (a) $\int_0^{\frac{\pi}{4}} \sin x\, dx$ (b) $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)\, dx$

   (c) $\int_0^1 \frac{1}{1+x^2} dx$ (d) $\int_1^2 \left(4x^3 - 5x^2 + 6x + 9\right) dx$

27.2 EVALUATION OF DEFINITE INTEGRAL BY SUBSTITUTION

The principal step in the evaluation of a definite integral is to find the related indefinite integral. In the preceding lesson we have discussed several methods for finding the indefinite integral. One of the important methods for finding indefinite integrals is the method of substitution. When we use substitution method for evaluation the definite integrals, like

$$\int_2^3 \frac{x}{1+x^2} dx, \quad \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx,$$

the steps could be as follows :

(i) Make appropriate substitution to reduce the given integral to a known form to integrate. Write the integral in terms of the new variable.

(ii) Integrate the new integrand with respect to the new variable.

(iii) Change the limits accordingly and find the difference of the values at the upper and lower limits.

Note : If we don't change the limit with respect to the new variable then after integrating resubstitute for the new variable and write the answer in original variable. Find the values of the answer thus obtained at the given limits of the integral.

Example 27.6 Evaluate $\int_2^3 \frac{x}{1+x^2}\,dx$

Solution : Let $1+x^2 = t$

$2x\,dx = dt$ or $x\,dx = \frac{1}{2}dt$

When $x = 2, t = 5$ and $x = 3, t = 10$. Therefore, 5 and 10 are the limits when t is the variable.

Thus
$$\int_2^3 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_5^{10}\frac{1}{t}\,dt$$
$$= \frac{1}{2}\left[\log t\right]_5^{10}$$
$$= \frac{1}{2}\left[\log 10 - \log 5\right]$$
$$= \frac{1}{2}\log 2$$

Example 27.7 Evaluate the following :

(a) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$ (b) $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\sin^4\theta + \cos^4\theta}\,d\theta$ (c) $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos x}$

Solution : (a) Let $\cos x = t$ then $\sin x\,dx = -\,dt$

When $x = 0, t = 1$ and $x = \frac{\pi}{2}, t = 0$. As x varies from 0 to $\frac{\pi}{2}$, t varies from 1 to 0.

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_1^0 \frac{1}{1+t^2}\,dt = -\left[\tan^{-1} t\right]_1^0$$
$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

Notes

$$= -\left[0 - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}$$

(b) $$I = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\sin^4\theta + \cos^4\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\left(\sin^2\theta + \cos^2\theta\right)^2 - 2\sin^2\theta\cos^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1 - 2\sin^2\theta\cos^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta\, d\theta}{1 - 2\sin^2\theta\left(1 - \sin^2\theta\right)}$$

Let $\sin^2\theta = t$

Then $2\sin\theta\cos\theta\, d\theta = dt$ i.e. $\sin 2\theta\, d\theta = dt$

When $\theta = 0, t = 0$ and $\theta = \frac{\pi}{2}, t = 1$. As θ varies from 0 to $\frac{\pi}{2}$, the new variable t varies from 0 to 1.

$\therefore$

$$I = \int_0^1 \frac{1}{1 - 2t(1-t)}\, dt$$

$$= \int_0^1 \frac{1}{2t^2 - 2t + 1}\, dt$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{t^2 - t + \frac{1}{4} + \frac{1}{4}}\, dt$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\, dt$$

$$= \frac{1}{2} \cdot \frac{1}{\frac{1}{2}}\left[\tan^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$= \left[\tan^{-1} 1 - \tan^{-1}(-1)\right]$$

Notes

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

(c) We know that $\cos x = \dfrac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{1}{5 + 4\cos x}\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{5 + \dfrac{4\left(1 - \tan^2\left(\frac{x}{2}\right)\right)}{\left(1 + \tan^2\left(\frac{x}{2}\right)\right)}}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{9 + \tan^2\left(\frac{x}{2}\right)}\, dx \qquad (1)$$

Let $\tan \frac{x}{2} = t$

Then $\sec^2 \frac{x}{2} dx = 2dt$ when $x = 0$, $t = 0$, when $x = \frac{\pi}{2}$, $t = 1$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{5 + 4\cos x} dx = 2\int_0^1 \frac{1}{9 + t^2} dt \qquad \text{[From (1)]}$$

$$= \frac{2}{3}\left[\tan^{-1} \frac{t}{3}\right]_0^1 = \frac{2}{3}\left[\tan^{-1} \frac{1}{3}\right]$$

27.3 SOME PROPERTIES OF DEFINITE INTEGRALS

The definite integral of f (x) between the limits a and b has already been defined as

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{Where } \frac{d}{dx}[F(x)] = f(x),$$

where a and b are the lower and upper limits of integration respectively. Now we state below some important and useful properties of such definite integrals.

(i) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$ where a<c<b.

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(vii) $$\int_0^{2a} f(x)\,dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \end{cases}$$

(viii) $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{if } f(x) \text{ is an odd function of } x \\ 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function of } x \end{cases}$$

Many of the definite integrals may be evaluated easily with the help of the above stated properties, which could have been very difficult otherwise.

The use of these properties in evaluating definite integrals will be illustrated in the following examples.

Example 27.8 Show that

(a) $$\int_0^{\frac{\pi}{2}} \log|\tan x|\,dx = 0$$ (b) $$\int_0^{\pi} \frac{x}{1+\sin x}\,dx = \pi$$

Solution : (a) Let $$I = \int_0^{\frac{\pi}{2}} \log|\tan x|\,dx \quad \text{....(i)}$$

Using the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get

$$I = \int_0^{\frac{\pi}{2}} \log\left(\tan\left(\frac{\pi}{2}-x\right)\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\cot x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log (\tan x)^{-1} dx$$

$$= -\int_0^{\frac{\pi}{2}} \log \tan x \, dx$$

$$= -I$$

$\therefore \quad 2I = 0$ [Using (i)]

i.e. $\quad I = 0 \quad$ or $\quad \int_0^{\frac{\pi}{2}} \log |\tan x| dx = 0$

(b) $\quad \int_0^{\pi} \frac{x}{1+\sin x} dx$

Let $\quad I = \int_0^{\pi} \frac{x}{1+\sin x} dx$ (i)

$$\therefore \quad I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)} dx \quad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin x} dx$$

Adding (i) and (ii) (ii)

$$2I = \int_0^{\pi} \frac{x + \pi - x}{1+\sin x} dx = \pi \int_0^{\pi} \frac{1}{1+\sin x} dx$$

or $\quad 2I = \pi \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x} dx$

$$= \pi \int_0^{\pi} \left(\sec^2 x - \tan x \sec x \right) dx$$

$$= \pi [\tan x - \sec x]_0^{\pi}$$

$$= \pi [(\tan \pi - \sec \pi) - (\tan 0 - \sec 0)]$$

$$= \pi [0 - (-1) - (0 - 1)]$$

$$= 2\pi$$

$$I = \pi$$

Notes

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad \text{(b)} \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x \cos x}dx$$

ion : (a) Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ (i)

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx$$

(Using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$).

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$= \int_0^{\frac{\pi}{2}} 1.dx$$

$$= [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{\pi}{4}$$

i.e. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \frac{\pi}{4}$ (i)

Notes

Then $I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)} dx$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} + \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x + \cos x - \sin x}{1 + \sin x \cos x} dx$$

$$= 0$$

$\therefore$ $I = 0$

Example 27.10 Evaluate (a) $\int_{-a}^{a} \frac{xe^{x^2}}{1 + x^2} dx$ (b) $\int_{-3}^{3} |x + 1|\, dx$

Solution : (a) Here $f(x) = \frac{xe^{x^2}}{1 + x^2}$

$\therefore$ $f(-x) = -\frac{xe^{x^2}}{1 + x^2}$

$= -f(x)$

$\therefore$ $f(x)$ is an odd function of x.

$\therefore$ $\int_{-a}^{a} \frac{xe^{x^2}}{1 + x^2} dx = 0$

(b) $\int_{-3}^{3} |x + 1|\, dx$

$$|x + 1| = \begin{cases} x + 1, \text{if } x \geq -1 \\ -x - 1, \text{if } x < -1 \end{cases}$$

$\therefore$ $\int_{-3}^{3} |x + 1|\, dx = \int_{-3}^{-1} |x + 1|\, dx + \int_{-1}^{3} |x + 1|\, dx$, using property (iii)

$$= \int_{-3}^{-1} (-x-1)\,dx + \int_{-1}^{3} (x+1)\,dx$$

$$= \left[\frac{-x^2}{2} - x\right]_{-3}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{3}$$

$$= -\frac{1}{2} + 1 + \frac{9}{2} - 3 + \frac{9}{2} + 3 - \frac{1}{2} + 1 = 10$$

Example 27.11 Evaluate $\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx$

Solution : Let $I = \int_0^{\frac{\pi}{2}} \log(\sin x)\,dx$ (i)

Also $$I = \int_0^{\frac{\pi}{2}} \log\left[\sin\left(\frac{\pi}{2} - x\right)\right]dx,$$ [Using property (iv)]

$$= \int_0^{\frac{\pi}{2}} \log(\cos x)\,dx$$ (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} [\log(\sin x) + \log(\cos x)]\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin x \cos x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx - \int_0^{\frac{\pi}{2}} \log(2)\,dx$$ (iii)

$$= \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx - \frac{\pi}{2}\log 2$$

Notes

Again, let $I_1 = \int_0^{\frac{\pi}{2}} \log(\sin 2x)\,dx$

Put $2x = t \Rightarrow dx = \frac{1}{2}dt$

When $x = 0$, $t = 0$ and $x = \frac{\pi}{2}, t = \pi$

$$\therefore \quad I_1 = \frac{1}{2}\int_0^{\pi} \log(\sin t)\,dt$$

$$= \frac{1}{2}.2\int_0^{\frac{\pi}{2}} \log(\sin t)\,dt,$$ [using property (vi)]

$$= \frac{1}{2}.2\int_0^{\frac{\pi}{2}} \log(\sin x)\,dt$$ [using property (i)]

$\therefore \quad I_1 = I,$

Putting this value in (iii), we get [from (i)] .....(iv)

$$2I = I - \frac{\pi}{2}\log 2 \quad \Rightarrow \quad I = -\frac{\pi}{2}\log 2$$

Hence, $\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx = -\frac{\pi}{2}\log 2$

CHECK YOUR PROGRESS 27.2

Evaluate the following integrals :

1. $\int_0^1 xe^{x^2}dx$

2. $\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\sin x}$

3. $\int_0^1 \frac{2x+3}{5x^2+1}dx$

4. $\int_{-5}^{5} |x+2|\,dx$

5. $\int_0^2 x\sqrt{2-x}\,dx$

6. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x}dx$

7. $\int_0^{\frac{\pi}{2}} \log\cos x\,dx$

8. $\int_{-a}^{a} \frac{x^3e^{x^4}}{1+x^2}dx$

9. $\int_0^{\frac{\pi}{2}} \sin 2x \log\tan x\,dx$

10. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x+\cos x}dx$

27.4 APPLICATIONS OF INTEGRATION

Suppose that f and g are two continuous functions on an interval [a, b] such that $f(x) \le g(x)$ for $x \in [a, b]$ that is, the curve $y = f(x)$ does not cross under the curve $y = g(x)$ over [a, b]. Now the question is how to find the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by $x = a$ and $x = b$.

Again what happens when the upper curve $y = f(x)$ intersects the lower curve $y = g(x)$ at either the left hand boundary $x = a$, the right hand boundary $x = b$ or both?

27.4.1 Area Bounded by the Curve, x-axis and the Ordinates

Let AB be the curve $y = f(x)$ and CA, DB the two ordinates at $x = a$ and $x = b$ respectively. Suppose $y = f(x)$ is an increasing function of x in the interval $a \le x \le b$.

Let P (x, y) be any point on the curve and $Q(x + \delta x, y + \delta y)$ a neighbouring point on it. Draw their ordinates PM and QN.

Here we observe that as x changes the area (ACMP) also changes. Let

$$A = \text{Area (ACMP)}$$

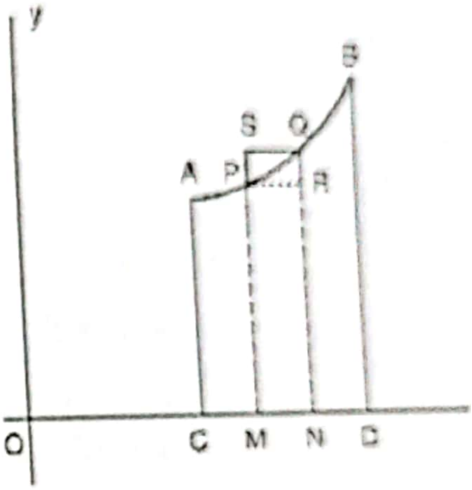

Fig.27.6

Then the area (ACNQ)= $A + \delta A$.

The area (PMNQ)=Area (ACNQ)−Area (ACMP)

$$= A + \delta A - A = \delta A.$$

Complete the rectangle PRQS. Then the area (PMNQ) lies between the areas of rectan[gles] PMNR and SMNQ, that is

δA lies between $y\,\delta x$ and $(y + \delta y)\delta x$

$\Rightarrow \quad \dfrac{\delta A}{\delta x}$ lies between y and $(y + \delta y)$

In the limiting case when $Q \to P, \delta x \to 0$ and $\delta y \to 0$.

$\therefore \quad \lim\limits_{\delta x \to 0} \dfrac{\delta A}{\delta x}$ lies between y and $\lim\limits_{\delta y \to 0} (y + \delta y)$

Notes

$$\int_a^b y\,dx = \int_a^b \frac{dA}{dx}\,dx = [A]_a^b$$

$$= (\text{Area when } x = b) - (\text{Area when } x = a)$$

$$= \text{Area (ACDB)} - 0$$

$$= \text{Area (ACDB)}.$$

Hence $\quad \text{Area (ACDB)} = \int_a^b f(x)\,dx$

The area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is

$$\int_a^b f(x)\,dx \text{ or } \int_a^b y\,dx$$

where $y = f(x)$ is a continuous single valued function and y does not change sign in the interval $a \le x \le b$.

Example 27.12 Find the area bounded by the curve $y = x$, x-axis and the lines $x = 0$, $x = 2$.

Solution : The given curve is $y = x$

$\therefore$ Required area bounded by the curve, x-axis and the ordinates $x = 0$, $x = 2$ (as shown in Fig. 27.7)

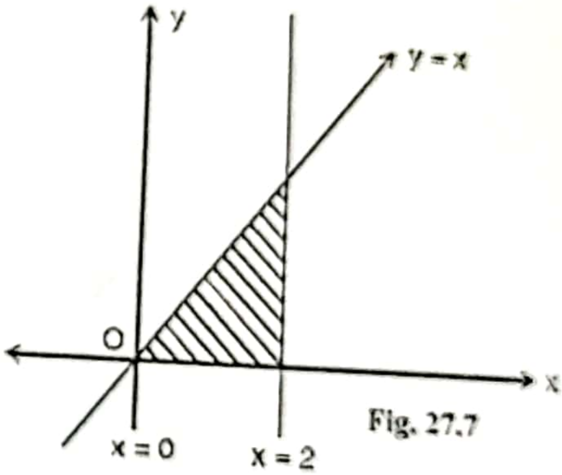

Fig. 27.7

is

$$\int_0^2 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^2$$

$$= 2 - 0 = 2 \text{ square units}$$

Example 27.13 Find the area bounded by the curve $y = e^x$, x-axis and the ordinates $x = 0$ and $x = a > 0$.

Solution : The given curve is $y = e^x$.

$\therefore$ Required area bounded by the curve, x-axis and the ordinates $x = 0$, $x = a$ is

$$\int_0^a e^x\,dx$$

$$= \left[e^x\right]_0^a$$

$$= (e^a - 1) \text{ square units}$$

Notes

Example 27.14 Find the area bounded by the curve $y = c\cos\left(\frac{x}{c}\right)$, x-axis and the ordinates $x = 0, x = a, \quad 2a \leq c\pi$.

Solution : The given curve is $y = c\cos\left(\frac{x}{c}\right)$

$$\therefore \quad \text{Required area} = \int_0^a y\,dx$$

$$= \int_0^a c\cos\left(\frac{x}{c}\right) dx$$

$$= c^2\left[\sin\left(\frac{x}{c}\right)\right]_0^a$$

$$= c^2\left(\sin\left(\frac{a}{c}\right) - \sin 0\right)$$

$$= c^2 \sin\left(\frac{a}{c}\right) \text{ square units}$$

Example 27.15 Find the area enclosed by the circle $x^2 + y^2 = a^2$, and x-axis in the first quadrant.

Solution : The given curve is $x^2 + y^2 = a^2$, which is a circle whose centre and radius are (0, 0) and a respectively. Therefore, we have to find the area enclosed by the circle $x^2 + y^2 = a^2$, the x-axis and the ordinates x = 0 and x = a.

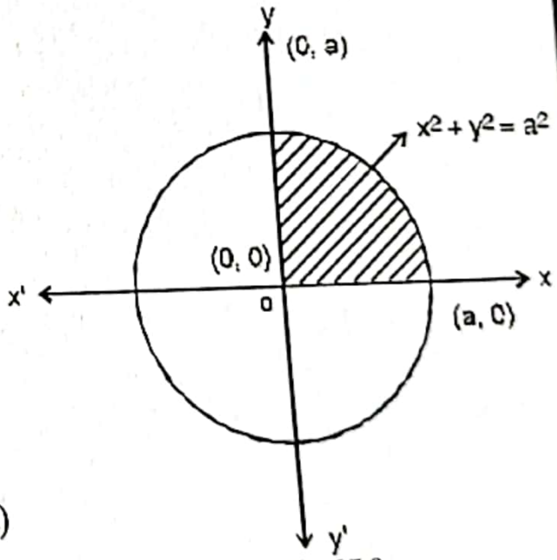

Fig. 27.8

$$\therefore \quad \text{Required area} = \int_0^a y\,dx$$

$$= \int_0^a \sqrt{a^2 - x^2}\,dx,$$

($\because$ y is positive in the first quadrant)

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 0 + \frac{a^2}{2}\sin^{-1} 1 - 0 - \frac{a^2}{2}\sin^{-1} 0$$

$$= \frac{a^2}{2}\cdot\frac{\pi}{2} \quad \left(\because \sin^{-1} 1 = \frac{\pi}{2}, \sin^{-1} 0 = 0\right)$$

$$= \frac{\pi a^2}{4} \text{ square units}$$

Example 27.16 Find the area bounded by the x-axis, ordinates and the following curves :

(i) $xy = c^2, x = a, x = b, \quad a > b > 0$

(ii) $y = \log_e x, x = a, x = b, \quad b > a > 1$

Solution : (i) Here we have to find the area bounded by the x-axis, the ordinates $x = a, x = b$ and the curve

$$xy = c^2 \qquad \text{or} \qquad y = \frac{c^2}{x}$$

$$\therefore \quad \text{Area} = \int_b^a y\,dx \qquad (\because a > b \text{ given})$$

$$= \int_b^a \frac{c^2}{x}\,dx$$

$$= c^2 [\log x]_b^a$$

$$= c^2 (\log a - \log b)$$

$$= c^2 \log\left(\frac{a}{b}\right)$$

(ii) Here $\quad y = \log_e x$

$$\therefore \quad \text{Area} = \int_a^b \log_e x\,dx, \qquad (\because b > a > 1)$$

$$= [x \log_e x]_a^b - \int_a^b x \cdot \frac{1}{x}\,dx$$

$$= b\log_e b - a\log_e a - \int_a^b dx$$

$$= b\log_e b - a\log_e a - [x]_a^b$$

$$= b\log_e b - a\log_e a - b + a$$

$$= b(\log_e b - 1) - a(\log_e a - 1)$$

$$= b\log_e\left(\frac{b}{e}\right) - a\log_e\left(\frac{a}{e}\right) \qquad (\because \log_e e = 1)$$

CHECK YOUR PROGRESS 27.3

1. Find the area bounded by the curve $y = x^2$, x-axis and the lines $x = 0, x = 2$.
2. Find the area bounded by the curve $y = 3x$, x-axis and the lines $x = 0$ and $x = 3$.

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Find the area bounded by the curve $y = e^{2x}$, x-axis and the ordinates $x = 0, x = a, a > 0$.

Find the area bounded by the x-axis, the curve $y = c\sin\left(\frac{x}{c}\right)$ and the ordinates $x = 0$ and $x = a$, $2a \le c\pi$.

27.4.2. Area Bounded by the Curve x = f (y) between y-axis and the Lines y = c, y = d

Let AB be the curve $x = f(y)$ and let CA, DB be the abscissae at $y = c$, $y = d$ respectively.

Let P (x, y) be any point on the curve and let $Q(x + \delta x,\ y + \delta y)$ be a neighbouring point on it. Draw PM and QN perpendiculars on y-axis from P and Q respectively. As y changes, the area (ACMP) also changes and hence clearly a function of y. Let A denote the area (ACMP), then the area (ACNQ) will be $A + \delta A$.

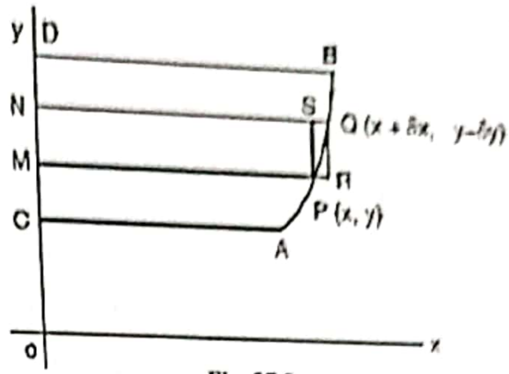

Fig. 27.9

The area (PMNQ) = Area (ACNQ) − Area (ACMP)= $A + \delta A - A = \delta A$.

Complete the rectangle PRQS. Then the area (PMNQ) lies between the area (PMNS) and the area (RMNQ), that is,

δA lies between $x\,\delta y$ and $(x + \delta x)\delta y$

$\Rightarrow \quad \frac{\delta A}{\delta y}$ lies between x and $x + \delta x$

In the limiting position when $Q \to P$, $\delta x \to 0$ and $\delta y \to 0$.

$\therefore \quad \lim\limits_{\delta y \to 0} \frac{\delta A}{\delta y}$ lies between x and $\lim\limits_{\delta x \to 0} (x + \delta x)$

$$\Rightarrow \quad \frac{dA}{dy} = x$$

Integrating both sides with respect to y, between the limits c to d, we get

$$\int_c^d x\,dy = \int_c^d \frac{dA}{dy}\cdot dy$$

$$= [A]_c^d$$

$$= \text{(Area when } y = d) - \text{(Area when } y = c)$$

$$= \text{Area (ACDB)} - 0$$

$$= \text{Area (ACDB)}$$

Hence area $\quad \text{(ACDB)} = \int_c^d x\,dy = \int_c^d f(y)\,dy$

Notes

The area bounded by the curve $x = f(y)$, the y-axis and the lines $y = c$ and $y = d$ is

$$\int_c^d x\,dy \quad \text{or} \quad \int_c^d f(y)\,dy$$

where $x = f(y)$ is a continuous single valued function and x does not change sign in the interval $c \le y \le d$.

Example 27.17 Find the area bounded by the curve $x = y$, y-axis and the lines $y = 0, y = 3$.

Solution : The given curve is $x = y$.

$\therefore$ Required area bounded by the curve, y-axis and the lines $y = 0$, $y = 3$ is

$$= \int_0^3 x\,dy$$

$$= \int_0^3 y\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^3$$

$$= \frac{9}{2} - 0$$

$$= \frac{9}{2} \text{ square units}$$

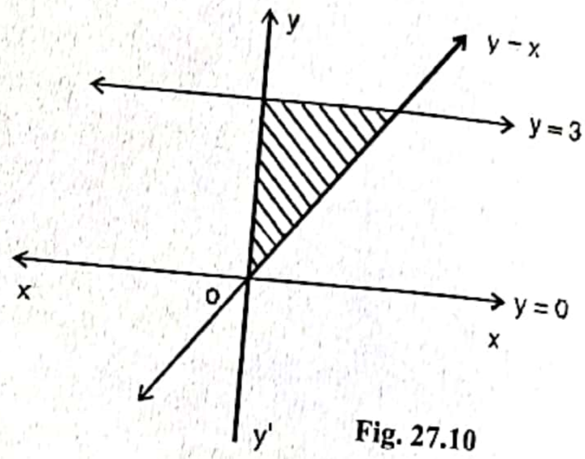

Fig. 27.10

Example 27.18 Find the area bounded by the curve $x = y^2$, y = axis and the lines $y = 0$, $y = 2$.

Solution : The equation of the curve is $x = y^2$

$\therefore$ Required area bounded by the curve, y-axis and the lines $y = 0$, $y = 2$

$$= \int_0^2 y^2 dy = \left[\frac{y^3}{3}\right]_0^2$$

$$= \frac{8}{3} - 0$$

$$= \frac{8}{3} \text{ square units}$$

Example 27.19 Find the area enclosed by the circle $x^2 + y^2 = a^2$ and y-axis in the first quadrant.

Solution : The given curve is $x^2 + y^2 = a^2$, which is a circle whose centre is (0, 0) and radius a. Therefore, we have to find the area enclosed by the circle $x^2 + y^2 = a^2$, the y-axis and the abscissae $y = 0, y = a$.

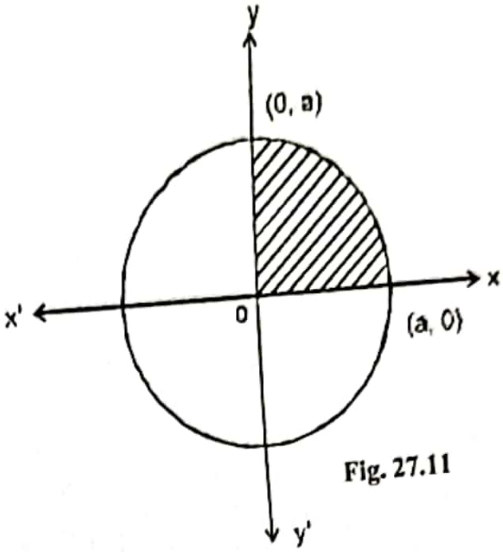

Fig. 27.11

$$\therefore \quad \text{Required area} = \int_0^a x \, dy$$

$$= \int_0^a \sqrt{a^2 - y^2} \, dy$$

(because x is positive in first quadrant)

$$= \left[\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right) \right]_0^a$$

$$= 0 + \frac{a^2}{2}\sin^{-1}1 - 0 - \frac{a^2}{2}\sin^{-1}0$$

$$= \frac{\pi a^2}{4} \text{ square units} \qquad \left(\because \sin^{-1}0 = 0, \sin^{-1}1 = \frac{\pi}{2} \right)$$

Note : The area is same as in Example 27.14, the reason is the given curve is symmetrical about both the axes. In such problems if we have been asked to find the area of the curve, without any restriction we can do by either method.

Example 27.20 Find the whole area bounded by the circle $x^2 + y^2 = a^2$.

Solution : The equation of the curve is $x^2 + y^2 = a^2$.

The circle is symmetrical about both the axes, so the whole area of the circle is four times the area os the circle in the first quadrant, that is,

Area of circle = 4 × area of OAB

$$= 4 \times \frac{\pi a^2}{4} \text{(From Example 27.15 and 27.19)} = \pi a^2$$

square units

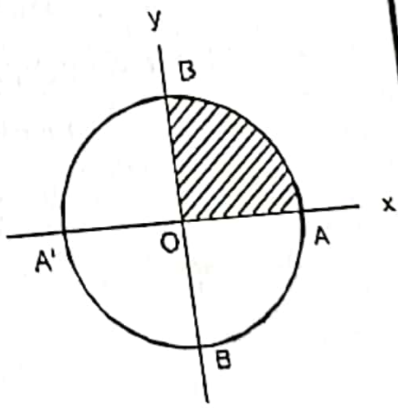

Fig. 27.12

Example 27.21 Find the whole area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution : The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Notes

The ellipse is symmetrical about both the axes and so the whole area of the ellipse is four times the area in the first quadrant, that is,

Whole area of the ellipse $= 4 \times$ area (OAB)

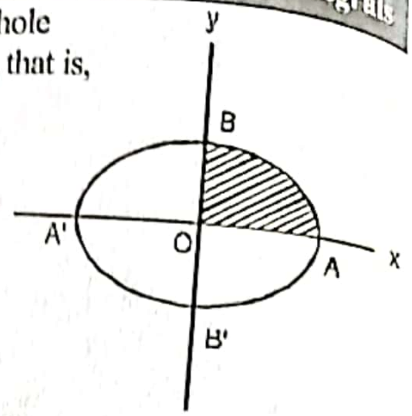

Fig.27.13

In the first quadrant,

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \quad \text{or} \quad y = \frac{b}{a}\sqrt{a^2 - x^2}$$

Now for the area (OAB), x varies from 0 to a

$$\therefore \quad \text{Area (OAB)} = \int_0^a y\,dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}1 - 0 - \frac{a^2}{2}\sin^{-1}0\right]$$

$$= \frac{ab\pi}{4}$$

Hence the whole area of the ellipse

$$= 4 \times \frac{ab\pi}{4}$$

$$= \pi ab. \text{ square units}$$

27.4.3 Area between two Curves

Suppose that f (x) and g (x) are two continuous and non-negative functions on an interval [a, b] such that $f(x) \geq g(x)$ for all $x \in [a, b]$ that is, the curve $y = f(x)$ does not cross under the curve $y = g(x)$ for $x \in [a, b]$. We want to find the area bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by $x = a$ and $x = b$.

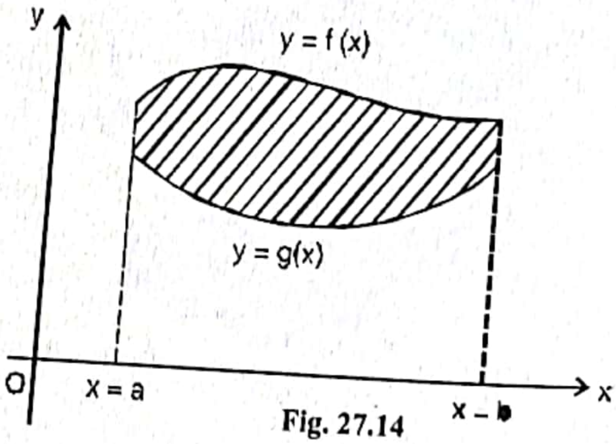

Fig. 27.14

Let A= [Area under $y = f(x)$] – [Area under $y = g(x)$] .....(1)

Now using the definition for the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$, we have

Area under

$$y = f(x) = \int_a^b f(x)\,dx \qquad .....(2)$$

Similarly, Area under $y = g(x) = \int_a^b g(x)\,dx \qquad .....(3)$

Using equations (2) and (3) in (1), we get

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx \qquad .....(4)$$

What happens when the function g has negative values also? This formula can be extended by translating the curves f (x) and g (x) upwards until both are above the x-axis. To do this let-m be the minimum value of g (x) on [a, b] (see Fig. 27.15).

Since $\quad g(x) \geq -m \quad \Rightarrow \quad g(x) + m \geq 0$

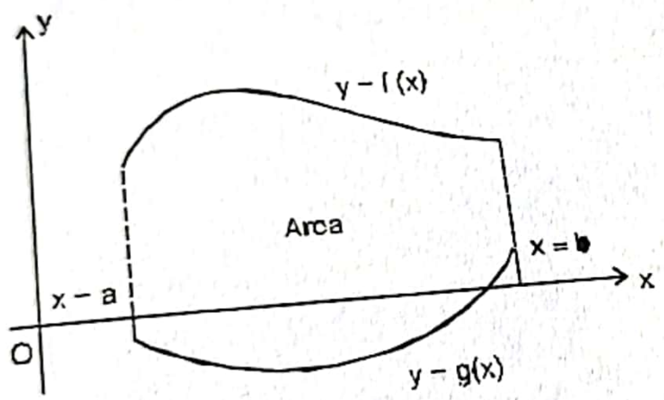

Fig. 27.15

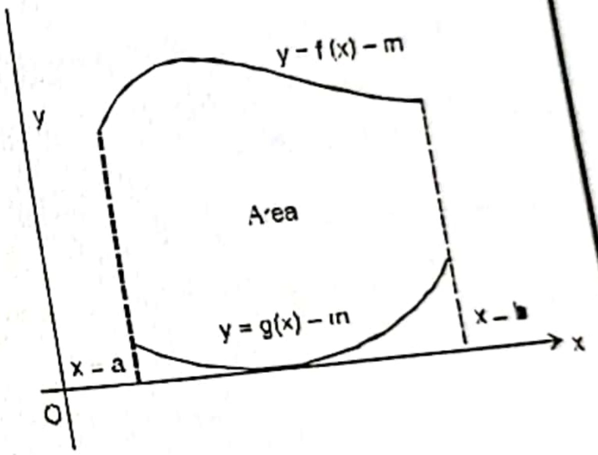

Fig. 27.16

Now, the functions $g(x) + m$ and $f(x) + m$ are non-negative on [a, b] (see Fig. 27.16). It is intuitively clear that the area of a region is unchanged by translation, so the area A between f and g is the same as the area between $g(x) + m$ and $f(x) + m$. Thus,

$A = [$ area under $y = [f(x) + m]] - [$area under $y = [g(x) + m]] \qquad .....(5)$

Now using the definitions for the area bounded by the curve y = f (x), x-axis and the ordinates x = a and x = b, we have ....

$$\text{Area under } y = f(x) + m = \int_a^b [f(x) + m]\,dx$$

Notes

and Area under $y = g(x) + m = \int_a^b [g(x) + m]\,dx$ (7)

The equations (6), (7) and (5) give

$$A = \int_a^b [f(x) + m]\,dx - \int_a^b [g(x) + m]\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$

which is same as (4) Thus,

If f (x) and g (x) are continuous functions on the interval [a, b], and $f(x) \geq g(x), \forall x \in [a, b]$, then the area of the region bounded above by y = f (x), below by y = g (x), on the left by x = a and on the right by x = b is

$$= \int_a^b [f(x) - g(x)]\,dx$$

Example 27.22 Find the area of the region bounded above by y = x + 6, bounded below by $y = x^2$, and bounded on the sides by the lines x = 0 and x = 2.

Solution : $y = x + 6$ is the equation of the straight line and $y = x^2$ is the equation of the parabola which is symmetric about the y-axis and origin the vertex. Also the region is bounded by the lines $x = 0$ and $x = 2$.

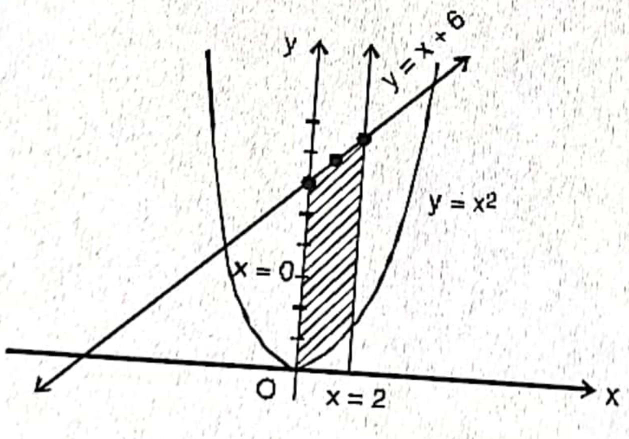

Fig. 27.17

Thus,

$$A = \int_0^2 (x + 6)\,dx - \int_0^2 x^2\,dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2$$

$$= \frac{34}{3} - 0$$

(7)

$$= \frac{34}{3} \text{ square units}$$

If the curves intersect then the sides of the region where the upper and lower curves intersect reduces to a point, rather than a vertical line segment.

Example 27.23 Find the area of the region enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution : We know that $y = x^2$ is the equation of the parabola which is symmetric about the y-axis and vertex is origin and $y = x + 6$ is the equation of the straight line which makes an angle 45° with the x-axis and having the intercepts of -6 and 6 with the x and y axes respectively. (See Fig. 27.18).

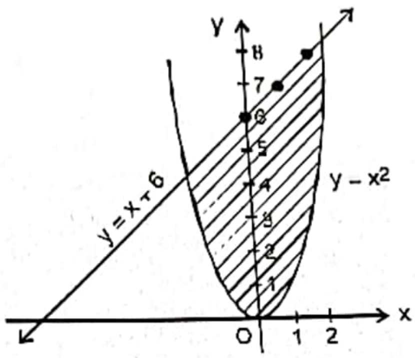

Fig. 27.18

A sketch of the region shows that the lower boundary is $y = x^2$ and the upper boundary is $y = x + 6$. These two curves intersect at two points, say A and B. Solving these two equations we get

$$x^2 = x + 6 \quad \Rightarrow \quad x^2 - x - 6 = 0$$

$$\Rightarrow \quad (x-3)(x+2) = 0 \quad \Rightarrow \quad x = 3, -2$$

When $x = 3$, $y = 9$ and when $x = -2$, $y = 4$

$$\therefore \text{ The required area } = \int_{-2}^{3}\left[(x+6) - x^2\right]dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$= \frac{27}{2} - \left(-\frac{22}{3}\right)$$

$$= \frac{125}{6} \text{ square units}$$

Example 27.24 Find the area of the region enclosed between the curves $y = x^2$ and $y = x$.

Solution : We know that $y = x^2$ is the equation of the parabola which is symmetric about the

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Notes

y-axis and vertex is origin. $y = x$ is the equation of the straight line passing through the origin and making an angle of 45° with the x-axis (see Fig. 27.19).

A sketch of the region shows that the lower boundary is $y = x^2$ and the upper boundary is the line $y = x$. These two curves intersect at two points O and A. Solving these two equations, we get

$$x^2 = x$$

$$\Rightarrow \quad x(x-1) = 0$$

$$\Rightarrow \quad x = 0, 1$$

Here $f(x) = x, g(x) = x^2, a = 0$ and $b = 1$

Therefore, the required area

$$= \int_0^1 (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ square units}$$

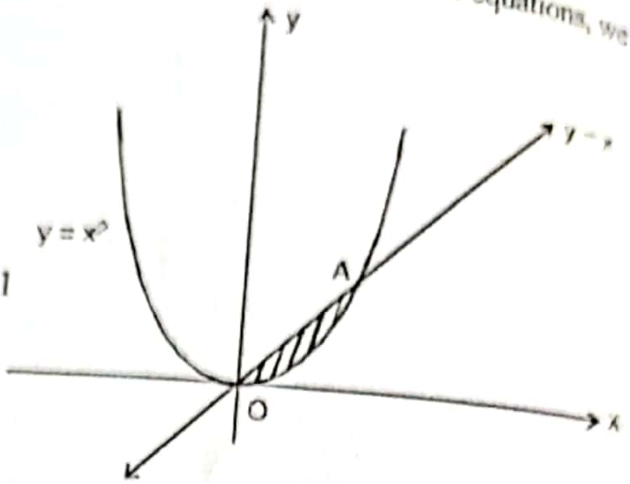

Fig. 27.19

Example 27.25 Find the area bounded by the curves $y^2 = 4x$ and $y = x$.

Solution : We know that $y^2 = 4x$ the equation of the parabola which is symmetric about the x-axis and origin is the vertex. $y = x$ is the equation of the straight line passing through origin and making an angle of 45° with the x-axis (see Fig. 27.20).

A sketch of the region shows that the lower boundary is $y = x$ and the upper boundary is $y^2 = 4x$. These two curves intersect at two points O and A. Solving these two equations, we get

$$\frac{y^2}{4} - y = 0$$

$$\Rightarrow \quad y(y-4) = 0$$

$$\Rightarrow \quad y = 0, 4$$

When $y = 0, x = 0$ and when $y = 4, x = 4$.

Here

$$f(x) = (4x)^{\frac{1}{2}}, g(x) = x, a = 0, b = 4$$

Therefore, the required area is

$$= \int_0^4 \left(2x^{\frac{1}{2}} - x\right) dx$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^4$$

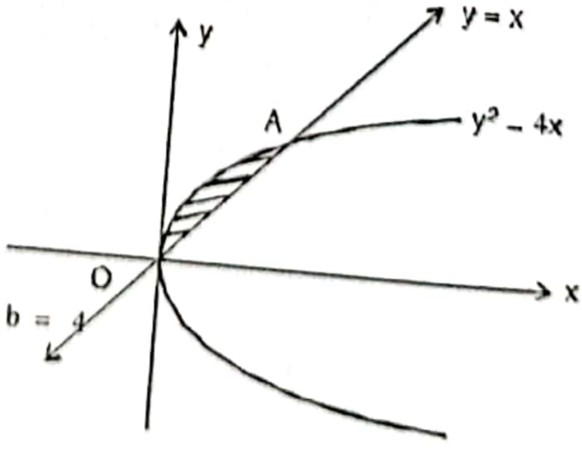

Fig. 27.20

$$= \frac{32}{3} - 8$$

$$= \frac{8}{3} \quad \text{square units}$$

Example 27.26 Find the area common to two parabolas $x^2 = 4ay$ and $y^2 = 4ax$.

Solution : We know that $y^2 = 4ax$ and $x^2 = 4ay$ are the equations of the parabolas, which are symmetric about the x-axis and y-axis respectively.

Also both the parabolas have their vertices at the origin (see Fig. 27.19).

A sketch of the region shows that the lower boundary is $x^2 = 4ay$ and the upper boundary is $y^2 = 4ax$. These two curves intersect at two points O and A. Solving these two equations, we have

$$\frac{x^4}{16a^2} = 4ax$$

$$\Rightarrow \quad x\left(x^3 - 64a^3\right) = 0$$

$$\Rightarrow \quad x = 0, 4a$$

Hence the two parabolas intersect at point (0, 0) and (4a, 4a).

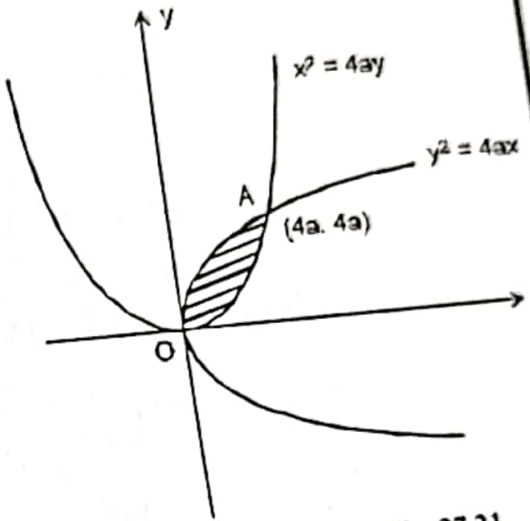

Fig. 27.21

Here $\quad f(x) = \sqrt{4ax}, g(x) = \frac{x^2}{4a}, a = 0 \text{ and } b = 4a$

Therefore, required area

$$= \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a} \right] dx$$

$$= \left[\frac{2.2\sqrt{a}x^{\frac{3}{2}}}{3} - \frac{x^3}{12a} \right]_0^{4a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16}{3}a^2 \text{ square units}$$

Notes

CHECK YOUR PROGRESS 27.4

1. Find the area of the circle $x^2 + y^2 = 9$

2. Find the area of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

3. Find the area of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

4. Find the area bounded by the curves $y^2 = 4ax$ and $y = \frac{x^2}{4a}$

5. Find the area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$.

6. Find the area enclosed by the curves $y = x^2$ and $y = x + 2$

LET US SUM UP

- If f is continuous in [a, b] and F is an anti derivative of f in [a, b], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- If f and g are continuous in [a, b] and c is a constant, then

(i) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

(ii) $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(iii) $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

- The area bounded by the curve $y = f(x)$, the x-axis and the ordinates

$$x = a, x = b \text{ is } \int_a^b f(x)\,dx \text{ or } \int_a^b y\,dx$$

where $y = f(x)$ is a continuous single valued function and y does not change sign in the interval $a \le x \le b$

If f (x) and g (x) are continuous functions on the interval [a, b] and $f(x) \geq g(x)$, for all $x \in [a,b]$, then the area of the region bounded above by y = f (x), below by y = g (x), on the left by x = a and on the right by x = b is

$$\int_a^b [f(x) - g(x)]dx$$

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

TERMINAL EXERCISE

Evaluate the following integrals (1 to 5) as the limit of sum.

1. $\int_a^b x\,dx$
2. $\int_a^b x^2\,dx$
3. $\int_a^b \sin x\,dx$
4. $\int_a^b \cos x\,dx$
5. $\int_0^2 (x^2+1)\,dx$

Evaluate the following integrals (6 to 25)

6. $\int_0^2 \sqrt{a^2 - x^2}\,dx$
7. $\int_0^{\frac{\pi}{2}} \sin 2x\,dx$
8. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\,dx$
9. $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$
10. $\int_0^1 \sin^{-1} x\,dx$
11. $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$
12. $\int_3^4 \frac{1}{x^2-4}\,dx$
13. $\int_0^{\pi} \frac{1}{5+3\cos\theta}\,d\theta$
14. $\int_0^{\frac{\pi}{4}} 2\tan^3 x\,dx$
15. $\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$
16. $\int_0^2 x\sqrt{x+2}\,dx$
17. $\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\cos^5\theta\,d\theta$

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18. $\int_0^{\pi} x \log \sin x \, dx$ 19. $\int_0^{\pi} \log(1+\cos x)\,dx$ 20. $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$

21. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}\,dx$ 22. $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$ 23. $\int_0^{\frac{\pi}{8}} \sin^5 2x \cos 2x \, dx$

24. $\int_0^{2} x\left(x^2+1\right)^3 dx$

ANSWERS

CHECK YOUR PROGRESS 27.1

1. $\frac{35}{2}$ 2. $e - \frac{1}{e}$

3. (a) $\frac{\sqrt{2}-1}{\sqrt{2}}$ (b) 2 (c) $\frac{\pi}{4}$ (d) $\frac{64}{3}$

CHECK YOUR PROGRESS 27.2

1. $\frac{e-1}{2}$ 2. $\frac{2}{3}\tan^{-1}\frac{1}{3}$ 3. $\frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$

4. 29 5. $\frac{24\sqrt{2}}{15}$ 6. $\frac{\pi}{4}$ 7. $-\frac{\pi}{2}\log 2$

8. 0 9. 0 10. $\frac{1}{2}\left[\frac{\pi}{2} - \log 2\right]$

CHECK YOUR PROGRESS 27.3

1. $\frac{8}{3}$ sq. units 2. $\frac{27}{2}$ sq. units 3. $\frac{e^{2a}-1}{2}$ sq. units

4. $c^2\left(1 - \cos\frac{a}{c}\right)$

CHECK YOUR PROGRESS 27.4

1. 9π sq. units 2. 6π sq. units 3. 20π sq. units

4. $\frac{16}{3}a^2$ sq. units 5. $\frac{16}{3}$ sq. units 6. $\frac{9}{2}$ sq. units

TERMINAL EXERCISE

1. $\frac{b^2 - a^2}{2}$ 2. $\frac{b^3 - a^3}{3}$ 3. $\cos a - \cos b$

4. $\sin b - \sin a$ 5. $\frac{14}{3}$ 6. $\frac{\pi a^2}{4}$

7. 1

8. $\frac{1}{2}\log 2$

9. $\frac{\pi}{4}$

10. $\frac{\pi}{2}-1$

11. $\frac{\pi}{2}$

12. $\frac{1}{4}\log\frac{5}{3}$

13. $\frac{\pi}{4}$

14. $1-\log 2$

15. $\frac{2}{3}$

16. $\frac{16}{15}\left(2+\sqrt{2}\right)$

17. $\frac{64}{231}$

18. $-\frac{\pi^2}{2}\log 2$

19. $-\pi\log 2$

20. $\frac{\pi^2}{4}$

21. $\frac{1}{\sqrt{2}}\log\left(1+\sqrt{2}\right)$

22. $\frac{\pi}{8}\log 2$

23. $\frac{1}{96}$

24. 78